

# UNIDIRECTIONAL EVOLUTION EQUATIONS OF DIFFUSION TYPE

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**ABSTRACT.** This paper is concerned with the uniqueness, existence, comparison principle and long-time behavior of solutions to the initial-boundary value problem for a *unidirectional diffusion equation*. The unidirectional evolution often appears in Damage Mechanics due to the strong irreversibility of crack propagation or damage evolution. The existence of solutions is proved in an  $L^2$ -framework by introducing a peculiar discretization of the unidirectional diffusion equation by means of variational inequities of obstacle type and by developing a regularity theory for such variational inequalities. The novel discretization argument will be also applied to prove the comparison principle as well as to investigate the long-time behavior of solutions.

## 1. INTRODUCTION

In Damage Mechanics, the *unidirectional evolution* is a significant feature of crack propagation models. Indeed, crack propagation is an irreversible phenomena, and particularly, cracks in a specimen or the damage of a material (e.g., microcracks which break or weaken bonds of microstructures) cannot disappear nor decrease. Hence if one introduces a phase parameter (see [33, 29, 43, 27, 30]) or an internal variable (see [12, 13]) which describes the crack growth or the damage accumulation, they are usually supposed to be unidirectional, i.e., nondecreasing or nonincreasing. Such unidirectional evolution processes are often described by PDEs involving the *positive-part function*,  $s \mapsto (s)_+ := s \vee 0 = \max\{s, 0\}$  for  $s \in \mathbb{R}$ . Moreover, in a phase field approach, the evolution of a phase parameter is governed by the gradient flow of an appropriate energy functional subject to some unidirectional constraint.

In order to find out mathematical features of such unidirectional evolutions and to develop mathematical devices to analyze them, in this paper, we shall treat, as a simplest case, the evolution of  $u = u(x, t)$  governed by the (fully nonlinear) PDE,

$$\partial_t u = (\Delta u + f)_+, \quad \text{for } x \in \Omega, \ t > 0, \quad (1)$$

where  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^n$  with  $n \in \mathbb{N}$ ,  $\partial_t u = \partial u / \partial t$ ,  $\Delta$  stands for the  $n$ -dimensional Laplacian,  $f = f(x, t)$  is a given function and  $(s)_+ := s \vee 0$  for  $s \in \mathbb{R}$ . More precisely, the main purpose of this paper is to prove the uniqueness, existence and comparison principle of *strong* solutions  $u = u(x, t)$  of the initial-boundary value problem for (1) and to reveal the asymptotic behavior of  $u = u(x, t)$  as  $t \rightarrow \infty$ .

Solutions of (1) entail *unidirectional* nature, more precisely, the non-decrease of  $u = u(x, t)$  in  $t$ , since the right-hand side of (1) is non-negative due to the presence of the positive part function. There appear various kinds of unidirectional evolutions in natural sciences and engineering fields (see, e.g., [21]). In particular, a phase field model for crack propagation in an elastic material was proposed with the aid of a unidirectional gradient flow by the second author and his collaborator in [27, 42, 43] (see also [6, 20]). Equation (1) can be regarded as a simplified equation of their model.

In mathematical points of view, (1) is classified as a fully nonlinear PDE, which is not fit for energy methods in general; however, by taking a (multi-valued) inverse function of the positive

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part function  $(\cdot)_+$ , (1) can be formulated as a sort of doubly nonlinear evolution equations,

$$\partial_t u + \alpha(\partial_t u) - \Delta u \ni f \quad \text{a.e. in } \Omega \times (0, \infty), \quad (2)$$

where  $\alpha$  is a (multi-valued) maximal monotone function in  $\mathbb{R}$  given by  $\alpha(0) = (-\infty, 0]$  and  $\alpha(s) = \{0\}$  for any  $s > 0$  with the domain  $D(\alpha) = [0, \infty)$  (see Remark 2.3 and Section 4 below for more details). Equation (2) is fitter for energy methods and monotone techniques. On the other hand, in view of the  $L^2$ -theory of evolution equations, two operators  $v \mapsto \alpha(v(\cdot))$  and  $v \mapsto -\Delta v$  (defined for  $v \in L^2(\Omega)$ ) are unbounded in  $L^2(\Omega)$ , and hence, it is more delicate to establish a priori estimates for proving the existence of strong solutions, as compared with standard equations without unidirectional constraints, e.g., the classical and nonlinear diffusion equations.

The nonlinear PDE (2) may fall within the frame of abstract doubly nonlinear evolution equations in a Hilbert space  $H$  of the form

$$\partial\Psi(\partial_t u(t)) + \partial\Phi(u(t)) \ni f(t) \quad \text{in } H, \quad 0 < t < T, \quad (3)$$

where  $\partial\Psi$  and  $\partial\Phi$  denote the subdifferential operator of functionals  $\Psi : H \rightarrow (-\infty, \infty]$  and  $\Phi : H \rightarrow (-\infty, \infty]$ , respectively. In a thermodynamic approach to continuum mechanics,  $\Psi$  and  $\Phi$  are often referred to as a *dissipation functional* and an *energy functional*, respectively. To reduce (2) into the form (3), we set  $u(t) := u(\cdot, t)$  and particularly choose

$$H = L^2(\Omega), \quad \Psi(v) = \frac{1}{2} \int_{\Omega} |v|^2 dx + I_{[\cdot \geq 0]}(v), \quad \Phi(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx \quad \text{for } v \in H,$$

where  $I_{[\cdot \geq 0]}$  is the indicator function over the set  $\{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$ . Then we note that both subdifferentials  $\partial\Psi$  and  $\partial\Phi$  are unbounded in  $H$ .

Let us briefly review the previous studies on abstract doubly nonlinear evolution equations such as (3). Barbu [11] proved the existence of solutions for (3) with two unbounded operators  $\partial\Psi$  and  $\partial\Phi$  by using the elliptic-in-time regularization and by imposing the differentiability (in  $t$ ) of  $f$ . This result was generalized by Arai [7], Senba [40] and so on. In these papers, the term  $\partial\Psi(\partial_t u(t))$  is estimated by differentiating the equation and by testing it with  $\partial\Psi(\partial_t u(t))$ . Therefore the differentiability of  $f$  (more precisely,  $f \in W^{1,1}(0, T; H)$  by [7]) is essentially required, and moreover, some strong monotonicity condition (i.e., the so-called  $\partial\Psi$ -monotonicity) is also imposed on  $\partial\Phi$ . Similar methods of establishing a priori estimates are also used in individual studies on irreversible phase transition models (see, e.g., [14]). On the other hand, Colli and Visintin established an alternative approach to (3) in [18], where  $\partial\Psi$  is supposed to be bounded and coercive with linear growth instead of assuming the regularity assumption on  $f$  and the  $\partial\Psi$ -monotonicity of  $\partial\Phi$  (see also [17]). Their framework would be more flexible in view of applications to nonlinear PDEs and has been extensively applied to various types of doubly nonlinear problems. Moreover, their framework has been generalized in many directions, e.g., perturbation problems, long-time behaviors (see, e.g., [1, 36, 35], [37, Sect. 11], [41, 39, 34, 38, 2, 3, 4, 5]). However, due to the unboundedness of  $\partial\Psi$ , (2) seems to be beyond the scope of the latter approach. On the other hand, the former approach due to Barbu and Arai is applicable to (2); however, the regularity assumption  $f \in W^{1,1}(0, T; H)$  has to be assumed for proving the existence of solutions. Aso et al. [8, 9] also treated an irreversible phase transition system in a different fashion; however, the regularity condition  $f \in W^{1,2}(0, T; H)$  is also assumed there.

In this paper, we present a novel approach to (1) (or equivalently (2)) by introducing a peculiar discretization for (1) by means of variational inequalities of obstacle type. Moreover, by developing a regularity theory of such variational inequalities, we shall establish new a priori estimates for (1) (or (2)) without assuming the differentiability (in  $t$ ) of  $f$ . Such a relaxation of the regularity assumption on  $f$  may bring some advantage to consider perturbation problems, which will be left for a forthcoming paper. Moreover, the novel discretization method will be also applied to investigate the long-time behavior of solutions as well as to prove a comparison theorem. In particular, we shall provide uniform (in  $t$ ) estimates for solutions by employing the peculiar discretization scheme. Furthermore, some variational inequality of obstacle type will play a crucial role in asymptotic analysis; indeed, it will turn out that every solution will converge to

the unique solution  $z = z(x)$  of a variational inequality of obstacle type involving the initial data as an obstacle function from below under suitable assumptions. Here it is worth mentioning that the limit  $z$  of the solution  $u = u(x, t)$  depends on its initial data  $u_0$ ; indeed, one can construct different limits of solutions for different initial data. On the other hand, the  $\omega$ -limit set of each solution must be singleton.

The organization of this paper is as follows. The initial-boundary value problem for (1) and basic notions (e.g., strong solution) and assumptions are formulated, and main results of this paper will be stated in the next section. Section 3 is devoted to establishing a regularity theory as well as to verifying a comparison theorem for variational inequalities of obstacle type. In Section 4, we discuss a reduction of (1) to an evolution equation of doubly nonlinear type in  $L^2(\Omega)$  and prove the uniqueness of solutions for the initial-boundary value problem. In Section 5, we carry out the backward-Euler time-discretization of (1) and construct a strong solution of (1) by proposing a new a priori estimate based on the regularity theory developed in Section 3. A comparison theorem for (1) is also proved here. The long-time behavior of solutions will be investigated in Section 6. In the last section, we shall discuss other equivalent formulations of solutions for (1).

**Notation.** For each normed space  $N$ , we denote by  $N'$  the dual space of  $N$  with the duality pairing  $\langle g, v \rangle_N := N' \langle g, v \rangle_N = g(v)$  for  $v \in N$  and  $g \in N'$ . For Banach spaces  $U$  and  $W$ , the set of all bounded linear operators from  $U$  to  $W$  is denoted by  $B(U, W)$ . Moreover, the set of all linear topological isomorphisms from  $U$  to  $W$  is denoted by  $\text{Isom}(U, W)$ , that is,  $A \in \text{Isom}(U, V)$  means that  $A$  is bijective from  $U$  to  $W$ ,  $A \in B(U, W)$  and  $A^{-1} \in B(W, U)$ . Furthermore,  $\mathcal{H}^k$  stands for the  $k$ -dimensional Hausdorff measure in  $\mathbb{R}^n$  for  $k = 1, 2, \dots, n$ . We also write  $a \vee b = \max\{a, b\}$  and  $a \wedge b = \min\{a, b\}$  for  $a, b \in \mathbb{R}$ . Finally,  $C$  denotes a non-negative constant independent of the elements of the corresponding space and set and may vary from line to line.

## 2. MAIN RESULTS

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$  with  $n \in \mathbb{N}$ . Let  $\Gamma$  be the boundary of  $\Omega$  such that  $\Gamma$  is the disjoint union of two subsets  $\Gamma_D$  and  $\Gamma_N$ , that is,

$$\Gamma_D \cup \Gamma_N = \Gamma, \quad \Gamma_D \cap \Gamma_N = \emptyset.$$

Moreover, assure that  $\Gamma_D$  is (relatively) open in  $\Gamma$ . Let  $\nu$  denote the outward-pointing unit normal vector on  $\Gamma$ . One of these two subsets may be empty. In such a case, the other set coincides with the whole of  $\Gamma$ . Main results of the present paper are concerned with the following initial-boundary value problem for a unidirectional evolution equation of diffusion type,

$$\partial_t u = (\Delta u + f)_+ \quad \text{in } Q := \Omega \times (0, \infty), \quad (4)$$

$$u = 0 \quad \text{on } \Gamma_D \times (0, \infty), \quad (5)$$

$$\partial_\nu u = 0 \quad \text{on } \Gamma_N \times (0, \infty), \quad (6)$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \quad (7)$$

where  $\partial_t = \partial/\partial t$ ,  $f = f(x, t)$  and  $u_0 = u_0(x)$  are given functions of class  $L^2_{loc}([0, \infty); L^2(\Omega))$  and  $L^2(\Omega)$ , respectively, and  $\partial_\nu u := \gamma_0(\nabla u) \cdot \nu$  denotes the normal derivative of  $u$ . Moreover,  $(\cdot)_+$  stands for the positive part function, i.e.,  $(s)_+ := s \vee 0$  for  $s \in \mathbb{R}$ . If  $\Gamma_D$  (resp.,  $\Gamma_N$ ) is empty, the corresponding boundary condition (5) (resp., (6)) is ignored.

**Remark 2.1.** By change of variable, one can reduce another unidirectional diffusion equation,

$$\partial_t u = (\Delta u + f)_- \quad \text{in } Q, \quad (8)$$

where  $(s)_- := s \wedge 0$  for  $s \in \mathbb{R}$ , to (4). Indeed, set  $v := -u$  and  $g := -f$ . Then (8) is transformed to

$$-\partial_t v = (-\Delta v - g)_- = -(\Delta v + g)_+,$$

whence  $v$  solves (4) with  $f$  replaced by  $g$ .

Let us start with defining *strong solutions* of (4)–(7). To this end, we set up notation. Let  $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$  denote the trace operator defined on  $H^1(\Omega)$  (throughout the paper, we may omit  $\gamma_0$  if no confusion can arise). Moreover, define

$$\begin{aligned} V &:= \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_D\}, \\ X &:= \{v \in H^2(\Omega) : \gamma_0(\nabla v) \cdot \nu = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_N\} \end{aligned}$$

equipped with the induced norms and inner products, i.e.,  $\|\cdot\|_V = \|\cdot\|_{H^1(\Omega)}$  and  $(\cdot, \cdot)_V = (\cdot, \cdot)_{H^1(\Omega)}$  for  $V$ ;  $\|\cdot\|_X = \|\cdot\|_{H^2(\Omega)}$  and  $(\cdot, \cdot)_X = (\cdot, \cdot)_{H^2(\Omega)}$  for  $X$ . Then  $V$  and  $X$  are closed subspaces of  $H^1(\Omega)$  and  $H^2(\Omega)$ , respectively; hence, they are Hilbert spaces. If either  $\Gamma_D$  or  $\Gamma_N$  is empty, the corresponding boundary condition specified in the definition of  $V$  or  $X$  above is ignored.

We are concerned with *strong solutions* of (4)–(7) defined by

**Definition 2.2** (Strong solution). *For  $T > 0$ , a function  $u \in C([0, T]; L^2(\Omega))$  is called a strong solution of (4)–(7) on  $[0, T]$ , if the following three conditions are satisfied:*

- (i)  $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; X \cap V)$ ,
- (ii) the equation  $\partial_t u = (\Delta u + f)_+$  holds a.e. in  $Q_T := \Omega \times (0, T)$ ,
- (iii) the initial condition  $u|_{t=0} = u_0$  holds a.e. in  $\Omega$ .

A function  $u \in C([0, \infty); L^2(\Omega))$  is called a strong solution of (4)–(7) on  $[0, \infty)$ , if for any  $T > 0$ , the restriction of  $u$  onto  $[0, T]$  is a strong solution of (4)–(7) on  $[0, T]$ .

**Remark 2.3.** One can further derive that  $u \in C([0, T]; V)$  from (i) by employing a chain-rule for convex functionals. See Lemma 4.4 in §4 below for more details.

We are now in position to state main results, whose proofs will be given in later sections. We begin with the uniqueness of solutions.

**Theorem 2.4** (Uniqueness). *Let  $T > 0$ ,  $u_0 \in V$  and  $f \in L^2(Q_T)$ . Then the strong solution of (4)–(7) on  $[0, T]$  is unique.*

To state our existence result, we shall introduce some assumptions for the domain  $\Omega$  and the boundary  $\Gamma_D, \Gamma_N$ . For  $\lambda \in \mathbb{R}$ , we define a mapping  $A_\lambda \in B(V, V')$  by

$$\langle A_\lambda u, v \rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv) \, dx \quad \text{for } u, v \in V. \quad (9)$$

It is well known that  $A_\lambda \in \text{Isom}(V, V')$  holds if  $\lambda > 0$ . Hence one can define  $u = A_\lambda^{-1}g$  for  $g \in L^2(\Omega)$  as the unique solution  $u$  of the elliptic problem in a weak form,

$$\int_\Omega (\nabla u \cdot \nabla v + \lambda uv) \, dx = \int_\Omega gv \, dx \quad \text{for all } v \in V,$$

(i.e.,  $A_\lambda u = g$  in  $V'$ ). Then we assume that

$$A_1^{-1}g \in H^2(\Omega) \quad \text{for all } g \in L^2(\Omega). \quad (10)$$

Condition (10) is often called an *elliptic regularity* condition and deeply related to the geometry of the domain and boundary conditions. Indeed, it holds true for smooth domains with a single boundary condition (i.e.,  $\Gamma_N = \emptyset$  or  $\Gamma_D = \emptyset$ ). However, it is more delicate to consider the validity of (10) for situations with nonsmooth domains or mixed boundary conditions. On the other hand, in order to take account of physical backgrounds of crack growth models and their numerical simulations, the regularity of the boundary may be at most Lipschitz continuous, and mixed boundary conditions seem to be natural as well. We shall give conditions equivalent to (10) in Proposition 3.6 below.

**Remark 2.5.** Let us exhibit a couple of examples of  $\Omega$ ,  $\Gamma_D$  and  $\Gamma_N$  for which the condition (10) is satisfied.

- (i) If  $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$  and  $\Gamma$  is of class  $C^{1,1}$ , then (10) is satisfied (see Theorem 2.2.2.3 and Theorem 2.2.2.5 of [25]).

- (ii) Let  $\Omega$  be convex. If  $\Gamma_D = \Gamma$  or  $\Gamma_D = \emptyset$ , then (10) is satisfied (see Theorem 3.2.1.2 and Theorem 3.2.1.3 of [25]).
- (iii) If  $n = 1$  or if  $\Omega$  is a rectangle in  $\mathbb{R}^2$  and  $\Gamma_D$  is a union of some of four edges of  $\Gamma$ , then (10) is satisfied. Indeed, since the weak solution  $u = A_1^{-1}g$  can be extended to an open neighborhood of  $\overline{\Omega}$  by reflection, it follows that  $u = A_1^{-1}g \in H^2(\Omega)$ .

Our existence result reads,

**Theorem 2.6** (Existence). *We suppose that the condition (10) holds true. Let  $T > 0$ ,  $u_0 \in X \cap V$  and  $f \in L^2(Q_T)$  be given. If there exists  $f^* \in L^2(\Omega)$  satisfying*

$$f(x, t) \leq f^*(x) \quad \text{a.e. in } Q_T, \quad (11)$$

*then there exists a strong solution  $u = u(x, t)$  to the problem (4)–(7) on  $[0, T]$ .*

**Remark 2.7** (Assumptions on  $f$ ). Condition (11) is weaker than  $f \in L^\infty(Q_T)$  or  $f \in W^{1,1}(0, T; L^2(\Omega))$  (cf. [7]). Indeed, if  $f \in W^{1,1}(0, T; L^2(\Omega))$ , then  $f^*(x) := f(x, 0) + \int_0^T |\partial_t f(x, t)| dt$  belongs to  $L^2(\Omega)$  and satisfies (11). On the other hand, (11) is stronger than  $f_+ := f \vee 0 \in L^\infty(0, T; L^2(\Omega))$ . In fact, (11) yields  $f_+ \in L^\infty(0, T; L^2(\Omega))$ . However, even if  $f_+ \in L^\infty(0, T; L^2(\Omega))$ , (11) might not hold true. One may easily find a counterexample, e.g.,  $f(x, t) = |x - t|^{-\alpha}$ ,  $\Omega = (0, 1)$ ,  $T = 1$  and  $0 < \alpha < 1/2$ .

Theorem 2.6 will be proved in Section 5. Our method of proof relies on the backward-Euler time-discretization of (4). The discretized equation will be rewritten as some variational inequalities of obstacle type. Moreover, developing a regularity theory for such variational inequalities, we shall obtain a new a priori estimate for discretized solutions. As a result, we shall construct a strong solution without assuming the differentiability of  $f$  in  $t$ , which is usually assumed in standard frameworks of doubly nonlinear evolution equations such as (32) (see, e.g., [11, 7, 40]).

Moreover, such a basic strategy of proving the existence result will be also applied to prove the following *comparison theorem* for strong solutions of (4)–(7):

**Theorem 2.8** (Comparison principle). *Let  $T > 0$  and suppose that (10) is satisfied. For each  $i = 1, 2$ , let  $u_0^i \in X \cap V$  and  $f^i \in L^2(Q_T)$  be such that there exists  $f^* \in L^2(\Omega)$  satisfying*

$$f^i(x, t) \leq f^*(x) \quad \text{a.e. in } Q_T.$$

*For  $i = 1, 2$ , let  $u^i = u^i(x, t)$  be the unique strong solution of (4)–(7) with  $u_0 = u_0^i$  and  $f = f^i$  on  $[0, T]$ . If  $u_0^1 \leq u_0^2$  a.e. in  $\Omega$  and  $f^1 \leq f^2$  a.e. in  $Q_T$ , then  $u^1 \leq u^2$  a.e. in  $Q_T$ .*

The peculiar discretization argument will be also employed to investigate the long-time behavior of solutions for (4)–(7). Furthermore, the comparison theorem stated above will play a crucial role to identify the limit of each solution  $u = u(x, t)$  as  $t \rightarrow \infty$ .

**Theorem 2.9** (Convergence of solutions as  $t \rightarrow \infty$ ). *Let  $u_0 \in X \cap V$  and assume that (10) holds and that*

- (H1)  $\mathcal{H}^{n-1}(\Gamma_D) > 0$ ;
- (H2) *there exists a function  $f_\infty \in L^2(\Omega)$  such that  $f - f_\infty$  belongs to  $L^2(0, \infty; L^2(\Omega))$ ;*
- (H3)  $f \in L^\infty(0, \infty; L^2(\Omega))$ , and (11) is satisfied.

*Then the unique solution  $u = u(x, t)$  of (4)–(7) on  $[0, \infty)$  converges to a function  $z = z(x) \in X \cap V$  strongly in  $V$  as  $t \rightarrow \infty$ . Moreover, the limit  $z$  satisfies*

$$z \geq u_0 \quad \text{and} \quad -\Delta z \geq f_\infty \quad \text{a.e. in } \Omega.$$

*In addition, if  $f(x, t) \leq f_\infty(x)$  for a.e.  $(x, t) \in Q$ , then the limit  $z$  coincides with the unique solution  $\bar{z} \in X \cap V$  of the following variational inequality:*

$$(VI)(u_0, f_\infty) \begin{cases} \bar{z} \in K_0(u_0) := \{v \in V : v \geq u_0 \text{ a.e. in } \Omega\}, \\ \int_\Omega \nabla \bar{z} \cdot \nabla (v - \bar{z}) dx \geq \int_\Omega f_\infty (v - \bar{z}) dx \quad \text{for all } v \in K_0(u_0). \end{cases}$$

**Remark 2.10.** Assumption (H1) is essentially required to ensure the convergence of the solution  $u = u(x, t)$  as  $t \rightarrow \infty$ . Indeed, suppose that  $\Gamma_D = \emptyset$  (i.e.,  $\Gamma_N = \Gamma$ ) and set  $u_0(x) \equiv 1$  and  $f(x, t) \equiv 1$ . The unique strong solution of (4)–(7) is given by

$$u(x, t) = 1 + t \quad \text{for } (x, t) \in Q,$$

and then,  $u(x, t)$  is divergent to  $\infty$  at each  $x \in \Omega$  as  $t \rightarrow \infty$ .

### 3. REGULARITY THEORY FOR VARIATIONAL INEQUALITIES OF OBSTACLE TYPE

In this section, based on the approach of Gustafsson [26], we revisit a regularity theory for variational inequalities of obstacle type. In classical literature on variational inequalities of obstacle type, the  $W^{2,p}(\Omega)$ -regularity of solutions is often obtained by using a penalization technique (see, e.g., [28, Chap. 4], [22]). On the other hand, Gustafsson [26] gave a simpler alternative proof by introducing an auxiliary variational inequality and by proving the coincidence of solutions for both problems. Let us also remark that, in previous results, it is assumed that  $W^{2,p}(\Omega) \subset C(\overline{\Omega})$  (namely,  $p > n/2$ ) in order to utilize the classical maximum principle for linear elliptic equations.

We shall establish a  $W^{2,p}(\Omega)$ -regularity result for variational inequalities of obstacle type equipped with a mixed boundary condition by properly modifying the argument of Gustafsson [26]. It is noteworthy that we do not assume that  $W^{2,p}(\Omega) \subset C(\overline{\Omega})$ , i.e.,  $p > n/2$ , as we employ Stampacchia's truncation technique instead of the classical maximum principle.

First, we shall set up notation. For  $\sigma \geq 0$ , let  $A := A_\sigma \in B(V, V')$  be defined as in (9). Define a symmetric bilinear form  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  associated with  $A$  by

$$a(u, v) := \langle Au, v \rangle_V = \int_{\Omega} (\nabla u \cdot \nabla v + \sigma uv) \, dx \quad \text{for } u, v \in V. \quad (12)$$

Throughout this section, we assume that

$$\sigma > 0 \quad \text{if } \mathcal{H}^{n-1}(\Gamma_D) = 0. \quad (13)$$

Under the condition (13), (by the Poincaré inequality for the case that  $\sigma = 0$  and  $\mathcal{H}^{n-1}(\Gamma_D) > 0$ ),  $a(\cdot, \cdot)$  turns out to be coercive on  $V \times V$ . Hence  $A$  is invertible, and  $A^{-1}$  belongs to  $B(V', V)$ . Let  $f \in V'$  and  $\psi \in V$  and define closed convex subsets  $K_0, K_1$  of  $V$  by

$$\begin{aligned} K_0 &:= \{v \in V : v \geq \psi \text{ a.e. in } \Omega\}, \\ K_1 &:= \{v \in V : Av \geq f \text{ in } V'\}. \end{aligned} \quad (14)$$

Here the inequality  $Av \geq f$  in  $V'$  means that  $\langle Av - f, \varphi \rangle_V \geq 0$  for all  $\varphi \in V$  satisfying  $\varphi \geq 0$  a.e. in  $\Omega$ . We also define functionals  $J, \hat{J}$  on  $V$  by

$$\begin{aligned} J(v) &:= \frac{1}{2}a(v, v) - \langle f, v \rangle_V \quad \text{for } v \in V, \\ \hat{J}(v) &:= \frac{1}{2}a(v, v) - \langle \hat{f}, v \rangle_V \quad \text{for } v \in V, \end{aligned} \quad (15)$$

where  $\hat{f} := A\psi \in V'$ . Then the following equivalence holds true:

**Proposition 3.1.** *Suppose (13) and let  $f \in V'$  and  $\psi \in V$ . Then the following five conditions for  $u \in V$  are equivalent to each other:*

- (a)  $u \in K_0$ ,  $J(u) \leq J(v)$  for all  $v \in K_0$ ,
- (b)  $u \in K_0$ ,  $a(u, v - u) \geq \langle f, v - u \rangle_V$  for all  $v \in K_0$ ,
- (c)  $u \in K_0 \cap K_1$ ,  $\langle Au - f, u - \psi \rangle_V = 0$ ,
- (d)  $u \in K_1$ ,  $a(u, v - u) \geq \langle \hat{f}, v - u \rangle_V$  for all  $v \in K_1$ ,
- (e)  $u \in K_1$ ,  $\hat{J}(u) \leq \hat{J}(v)$  for all  $v \in K_1$ .

Moreover, there exists a unique element  $u \in V$  satisfying all the conditions.



*Proof.* Since  $J$  is a coercive, continuous, strictly convex functional on the closed convex set  $K_0$ ,  $J$  admits a unique minimizer  $u$  over  $K_0$ . Hence  $u$  satisfies (a).

We shall prove the equivalence of the conditions (a)–(e). It is well known (see, e.g., [28]) that (a)  $\Leftrightarrow$  (b) and (d)  $\Leftrightarrow$  (e). So, let us here start with showing that (b)  $\Rightarrow$  (c). The condition (b) is equivalently rewritten by

$$u \in K_0, \quad \langle Au - f, v - u \rangle_V \geq 0 \quad \text{for all } v \in K_0. \quad (16)$$

For any  $\varphi \in V$  with  $\varphi \geq 0$  a.e. in  $\Omega$ , substituting  $v = u + \varphi \in K_0$  to (16), we have  $\langle Au - f, \varphi \rangle_V \geq 0$ , which yields that  $u \in K_1$ . On the other hand, substitute  $v = \psi \in K_0$  and  $v = 2u - \psi \in K_0$  to (16). Then one can obtain  $\langle Au - f, u - \psi \rangle_V = 0$ . Hence (c) holds.

To prove the inverse relation, (c)  $\Rightarrow$  (b), let  $u$  satisfy (c). For any  $v \in K_0$ , we see that

$$a(u, v - u) - \langle f, v - u \rangle_V = \langle Au - f, v - \psi \rangle_V - \langle Au - f, u - \psi \rangle_V.$$

Since the first term of the right-hand side is nonnegative (by  $u \in K_1$  and  $v \in K_0$ ) and the second term vanishes (by the equation of (c)), the condition (b) follows.

One may prove the equivalence between (c) and (d) in a similar fashion to the above. Firstly, suppose that  $u$  satisfies (c). For any  $v \in K_1$ , one finds that

$$\begin{aligned} a(u, v - u) - \langle \hat{f}, v - u \rangle_V &= \langle Av - Au, u \rangle_V - \langle Av - Au, \psi \rangle_V \\ &= \langle Av - f, u - \psi \rangle_V - \langle Au - f, u - \psi \rangle_V. \end{aligned}$$

Here we used the fact that  $\langle Aw, z \rangle_V = \langle Az, w \rangle_V$  for all  $w, z \in V$ . Noting that the right-hand side is non-negative by (c) and the fact that  $u \in K_0$  and  $v \in K_1$ , one can get (d). To check the inverse relation, we also rewrite (d) as

$$u \in K_1, \quad \langle Av - Au, u - \psi \rangle_V \geq 0 \quad \text{for all } v \in K_1. \quad (17)$$

For any  $\varphi \in L^2(\Omega)$  with  $\varphi \geq 0$  a.e. in  $\Omega$ , substituting  $v = u + A^{-1}\varphi \in K_1$  to (17), we have  $\langle \varphi, u - \psi \rangle_{L^2(\Omega)} \geq 0$ , which along with the arbitrariness of  $\varphi \geq 0$  implies  $u \in K_0$ . Moreover, let us also substitute  $v = A^{-1}f \in K_1$  and  $v = 2u - A^{-1}f \in K_1$  in (17). Then we obtain  $\langle Au - f, u - \psi \rangle_V = 0$ , whence (c) follows. Consequently, all the conditions (a)–(e) are equivalent.  $\square$

In the rest of this section, let  $p \in \mathbb{R}$  satisfy

$$1 < p < \infty, \quad p \geq \frac{2n}{n+2}. \quad (18)$$

Since the Hölder conjugate  $q := p/(p-1)$  of  $p$  satisfies  $q \leq 2^* := 2n/(n-2)$  if  $n \geq 3$ , and  $\Omega$  is a Lipschitz domain, by Sobolev's embedding theorem, the continuous embeddings  $V \hookrightarrow L^q(\Omega)$  and  $L^p(\Omega) \cong (L^q(\Omega))' \hookrightarrow V'$  hold true. We also note that  $W^{2,p}(\Omega)$  is continuously embedded in  $H^1(\Omega)$  by (18).

We further suppose that

$$f \in L^p(\Omega), \quad \psi \in V, \quad A\psi \in L^p(\Omega) \quad (19)$$

and introduce a closed (in  $V$ ) convex set  $K_2$  given by

$$K_2 := \{v \in V : f \leq Av \leq f \vee \hat{f} \text{ in } V'\} \subset K_1.$$

A main result of this section is stated as follows:

**Theorem 3.2.** *Suppose that (13), (18) and (19) are satisfied. Then each of the following conditions (f)–(h) is equivalent to the conditions (a)–(e) of Proposition 3.1:*

- (f)  $u \in K_2$ ,  $\hat{J}(u) \leq \hat{J}(v)$  for all  $v \in K_2$ ,
- (g)  $u \in K_2$ ,  $a(u, v - u) \geq \langle \hat{f}, v - u \rangle_V$  for all  $v \in K_2$ ,
- (h)  $u \in K_0 \cap K_2$ ,  $(Au - f)(u - \psi) = 0$  a.e. in  $\Omega$ .

We first prepare a couple of lemmas, which will be also used in later sections.

**Lemma 3.3.** *Let  $w \in H^1(\Omega)$  and set  $w_+(x) := (w(x))_+$  for  $x \in \Omega$ . Then  $w_+ \in H^1(\Omega)$  and  $\gamma_0(w_+) = (\gamma_0 w)_+ \mathcal{H}^{n-1}$ -a.e. on  $\Gamma$ .*

*Proof.* Set  $\Omega_+ := \{x \in \Omega : w(x) > 0\}$  and recall Theorem A.1. of [28] to observe that  $w_+ \in H^1(\Omega)$  and

$$w_+ = \begin{cases} w & \text{a.e. in } \Omega_+, \\ 0 & \text{a.e. in } \Omega \setminus \Omega_+, \end{cases} \quad \nabla w_+ = \begin{cases} \nabla w & \text{a.e. in } \Omega_+, \\ 0 & \text{a.e. in } \Omega \setminus \Omega_+. \end{cases} \quad (20)$$

Set  $W := C(\overline{\Omega}) \cap H^1(\Omega)$ . Since  $W$  is dense in  $H^1(\Omega)$ , there exists a sequence  $\{w_n\}$  in  $W$  such that  $w_n \rightarrow w$  strongly in  $H^1(\Omega)$  as  $n \rightarrow \infty$ . Noting that

$$\|(w_n)_+ - w_+\|_{L^2(\Omega)} \leq \|w_n - w\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (21)$$

we observe that  $(w_n)_+ \rightarrow w_+$  strongly in  $L^2(\Omega)$ . Applying (20) to  $w_n$ , we also have

$$\|(w_n)_+\|_{H^1(\Omega)}^2 = \|(w_n)_+\|_{L^2(\Omega)}^2 + \|\nabla(w_n)_+\|_{L^2(\Omega)}^2 \leq \|w_n\|_{L^2(\Omega)}^2 + \|\nabla w_n\|_{L^2(\Omega)}^2 = \|w_n\|_{H^1(\Omega)}^2.$$

Since  $\{w_n\}$  is bounded in  $H^1(\Omega)$ , so is  $\{(w_n)_+\}$ . Hence, one can extract a (non-relabeled) subsequence of  $\{n\}$  such that  $(w_n)_+ \rightarrow w_+$  weakly in  $H^1(\Omega)$ . Again from (20), we have

$$\begin{aligned} \|(w_n)_+\|_{H^1(\Omega)}^2 &= \|(w_n)_+\|_{L^2(\Omega)}^2 + (\nabla(w_n)_+, \nabla w_n)_{L^2(\Omega)} \\ &\rightarrow \|w_+\|_{L^2(\Omega)}^2 + (\nabla w_+, \nabla w)_{L^2(\Omega)} = \|w_+\|_{H^1(\Omega)}^2, \end{aligned}$$

which together with the uniform convexity of  $H^1(\Omega)$  also implies that  $(w_n)_+ \rightarrow w_+$  strongly in  $H^1(\Omega)$ . Since  $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$ , we particularly deduce that  $\gamma_0 w_n \rightarrow \gamma_0 w$  and  $\gamma_0(w_n)_+ \rightarrow \gamma_0 w_+$  strongly in  $L^2(\Gamma)$ . As in (21), one can verify that  $(\gamma_0 w_n)_+ \rightarrow (\gamma_0 w)_+$  strongly in  $L^2(\Gamma)$ . On the other hand, since  $w_n \in C(\overline{\Omega})$ , it is clear that

$$\gamma_0(w_n)_+ = (\gamma_0 w_n)_+ \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma. \quad (22)$$

Passing to the limit as  $n \rightarrow \infty$  in (22), we conclude that  $\gamma_0 w_+ = (\gamma_0 w)_+ \mathcal{H}^{n-1}$ -a.e. on  $\Gamma$ .  $\square$

**Lemma 3.4.** *If  $v_1, v_2 \in V$ , then  $v_1 \vee v_2 \in V$  and  $v_1 \wedge v_2 \in V$ .*

*Proof.* Applying Lemma 3.3 to  $w = \pm(v_1 - v_2)$ , we find that  $(v_1 - v_2)_+$  and  $(v_2 - v_1)_+$  belong to  $V$ . Hence we obtain  $v_1 \vee v_2 \in V$  and  $v_1 \wedge v_2 \in V$ , since  $v_1 \vee v_2 = v_2 + (v_1 - v_2)_+$  and  $v_1 \wedge v_2 = v_2 - (v_2 - v_1)_+$ .  $\square$

Let us move on to a proof of Theorem 3.2.

*Proof of Theorem 3.2.* It is obvious that (f)  $\Leftrightarrow$  (g). As in Proposition 3.1, one can uniquely choose  $u \in K_2$  which satisfies (f) and (g). Next, we shall prove the equivalence between (a)–(e) and (f), (g). Let  $u_1$  be the unique element of  $V$  satisfying (a)–(e) and let  $u_2$  be the unique element of  $V$  satisfying (f) and (g).

We claim that  $u_1 = u_2$ . Indeed, note that  $Au_2 \in L^p(\Omega)$  by  $u_2 \in K_2$ . Set  $w := u_2 - \psi \in V$  and  $h := Aw \in L^p(\Omega)$ . Since  $u_2$  satisfies (g), it follows that

$$\begin{aligned} 0 \leq a(u_2, v - u_2) - \langle \hat{f}, v - u_2 \rangle_V &= \langle Au_2 - A\psi, v - u_2 \rangle_V \\ &= \langle Av - Au_2, w \rangle_V \quad \text{for all } v \in K_2. \end{aligned} \quad (23)$$

We set a measurable set

$$N := \{x \in \Omega : w(x) < 0\}.$$

Define  $g \in L^p(\Omega)$  by

$$g(x) := \begin{cases} f(x) \vee \hat{f}(x) & \text{if } x \in N, \\ Au_2(x) & \text{if } x \in \Omega \setminus N. \end{cases}$$



Then by definition one can observe that  $A^{-1}g \in K_2$ . Hence substituting  $v = A^{-1}g$  to (23), we have

$$0 \leq \langle g - Au_2, w \rangle_V = \int_N \left( (f \vee \hat{f}) - Au_2 \right) w \, dx.$$

Since  $(f \vee \hat{f}) - Au_2 \geq 0$  and  $w < 0$  a.e. in  $N$ , one can derive the relation  $Au_2 = f \vee \hat{f}$  a.e. in  $N$ . It follows that

$$h = Au_2 - A\psi = (f \vee \hat{f}) - \hat{f} = (f - \hat{f})_+ \geq 0 \quad \text{a.e. in } N.$$

On the other hand, we recall that  $w \in V$  solves the equation  $Aw = h$  in  $V'$ , that is,

$$a(w, v) = \langle h, v \rangle_V \quad \text{for all } v \in V. \quad (24)$$

We define  $w_- := w \wedge 0$ . Then by Lemma 3.4 and Theorem A.1. of [28], we have

$$w_- \in V, \quad w_- = \begin{cases} w & \text{in } N, \\ 0 & \text{in } \Omega \setminus N, \end{cases} \quad \nabla w_- = \begin{cases} \nabla w & \text{a.e. in } N, \\ 0 & \text{a.e. in } \Omega \setminus N. \end{cases}$$

Substituting  $v = w_-$  into (24), we have

$$0 \leq a(w_-, w_-) = a(w, w_-) \stackrel{(24)}{=} \langle h, w_- \rangle_V = \int_N hw \, dx. \quad (25)$$

Since  $h \geq 0$  and  $w < 0$  a.e. in  $N$ , the right-hand side of (25) is non-positive. Hence we deduce that  $a(w_-, w_-) = 0$ , which along with the coercivity of  $a(\cdot, \cdot)$  implies that  $w_- = 0$  (i.e.,  $w \geq 0$ ) a.e. in  $\Omega$ . Therefore  $u_2$  belongs to  $K_0$ .

Substitute  $v = A^{-1}f \in K_2$  to the condition (23). Then we obtain

$$\langle Au_2 - f, u_2 - \psi \rangle_V \leq 0. \quad (26)$$

Moreover, noting that  $Au_2 - f \geq 0$  (by  $u_2 \in K_2$ ) and  $u_2 - \psi \geq 0$  (by  $u_2 \in K_0$ ), we derive  $\langle Au_2 - f, u_2 - \psi \rangle_V = 0$  by (26). Hence,  $u = u_2$  satisfies the condition (c). By uniqueness, we obtain  $u_1 = u_2$ . Thus we have proved that all the conditions (a)–(g) are equivalent.

Finally, we note that (h) immediately implies (c), since  $K_2 \subset K_1$ . Conversely, let  $u$  satisfy (c). Then  $u$  belongs to  $K_2$  by (f), and hence,  $Au \in L^p(\Omega)$  and

$$0 = \langle Au - f, u - \psi \rangle_V = \int_{\Omega} (Au - f)(u - \psi) \, dx.$$

Thus we obtain  $(Au - f)(u - \psi) = 0$  a.e. in  $\Omega$ , since  $Au \geq f$  and  $u \geq \psi$  a.e. in  $\Omega$ . Therefore (h) holds.  $\square$

Thanks to Theorem 3.2, for each solution  $u$  of the variational inequality of obstacle type,

$$u \in K_0, \quad a(u, v - u) \geq \langle f, v - u \rangle_V \quad \text{for all } v \in K_0, \quad (27)$$

we have obtained an additional information,  $u \in K_2$ . In order to more explicitly clarify the feature of the additional information, let us introduce the following assumption:

$$\begin{cases} A_1^{-1}g \in W^{2,p}(\Omega) & \text{for all } g \in L^p(\Omega) & \text{in case } \sigma > 0 \text{ or } p \leq 2^*, \\ A_1^{-1}g \in W^{2,\rho}(\Omega) & \text{for all } g \in L^\rho(\Omega) \text{ and } \rho \in [2^*, p] & \text{in case } \sigma = 0 \text{ and } p > 2^*, \end{cases} \quad (28)$$

where  $2^* = 2n/(n-2)_+$  (here we note that (10) is a special case of (28) with  $p = 2$ ). Then we find that

**Proposition 3.5.** *Assume that (28) holds. Then*

$$K_2 \subset W^{2,p}(\Omega).$$

*Proof.* Let  $v \in K_2$ . In case  $\sigma > 0$  or  $p \leq 2^*$  (hence  $V \hookrightarrow L^p(\Omega)$ ), since  $Av \in L^p(\Omega)$  and  $v \in V$ , one can check that  $v \in L^p(\Omega)$ . Hence  $A_1v = Av + (1 - \sigma)v \in L^p(\Omega)$ , which along with (28) gives  $v \in W^{2,p}(\Omega)$ . In case  $\sigma = 0$  and  $p > 2^*$ , noting that  $A_1v \in L^{2^*}(\Omega)$  as above, we find by (28) that  $v \in W^{2,2^*}(\Omega)$ . By iteration argument along with Sobolev's embeddings, one can finally conclude that  $v \in L^p(\Omega)$ . Therefore we deduce that  $v \in W^{2,p}(\Omega)$  as in the former case.  $\square$

Condition (28) can be regarded as *elliptic regularity* of weak solutions for the elliptic boundary value problem,

$$-\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u = 0 \text{ on } \Gamma_N,$$

and it holds true in many cases, e.g., smooth domains with  $\Gamma_N = \emptyset$  or  $\Gamma_D = \emptyset$  (see, e.g., [24]). However, the validity of (28) is more delicate, if  $\Omega$  is not smooth or mixed boundary conditions are imposed. So we here explicitly made the assumption.

To take account of boundary conditions, we further define a subspace of  $W^{2,p}(\Omega)$  by

$$X^p := \{v \in W^{2,p}(\Omega) : \gamma_0(\nabla v) \cdot \nu = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_N\}.$$

Prior to stating a regularity result for (27), we prepare a proposition, which provides equivalent forms of the assumption (28).

**Proposition 3.6.** *Under the assumption (18), the operator  $A_\lambda|_{X^p \cap V}$  restricted onto  $X^p \cap V$  is injective and bounded linear from  $X^p \cap V$  into  $L^p(\Omega)$ , and it coincides with the operator  $-\Delta + \lambda$ , where  $\Delta$  means the Laplace operator from  $D(\Delta) = X^p \cap V$  into  $L^p(\Omega)$ , that is, the Laplacian equipped with the Dirichlet and Neumann boundary conditions on  $\Gamma_D$  and  $\Gamma_N$ , respectively, in a strong form.*

Moreover, the following conditions for  $\Omega$  and  $\Gamma_D, \Gamma_N$  are equivalent to each other:

- (i) there exists  $\lambda > 0$  such that  $A_\lambda^{-1}g \in W^{2,p}(\Omega)$  for all  $g \in L^p(\Omega)$ ;
- (ii) for any  $\lambda > 0$ , it holds that  $A_\lambda^{-1}g \in W^{2,p}(\Omega)$  for all  $g \in L^p(\Omega)$ ;
- (iii) there exists  $\lambda > 0$  such that  $A_\lambda^{-1}g \in X^p$  for all  $g \in L^p(\Omega)$ , and  $(-\Delta + \lambda) \in \text{Isom}(X^p \cap V, L^p(\Omega))$ ;
- (iv) for any  $\lambda > 0$ , it holds that  $A_\lambda^{-1}g \in X^p$  for all  $g \in L^p(\Omega)$ , and  $(-\Delta + \lambda) \in \text{Isom}(X^p \cap V, L^p(\Omega))$ .

*Proof.* Denote  $B_\lambda := A_\lambda|_{X^p \cap V}$  for  $\lambda > 0$ . Then, for  $u \in X^p \cap V$ , we observe by Green's formula, which is valid for Lipschitz domains, that

$$\langle B_\lambda u, v \rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv) \, dx = \int_\Omega (-\Delta u + \lambda u) v \, dx \quad \text{for all } v \in V \cap W^{1,q}(\Omega),$$

which implies that  $B_\lambda u = -\Delta u + \lambda u$  and  $B_\lambda \in B(X^p \cap V, L^p(\Omega))$ , since  $V \cap W^{1,q}(\Omega)$  is dense in  $L^q(\Omega)$ . Thus we obtain  $B_\lambda = (-\Delta + \lambda)$ . Moreover,  $B_\lambda$  is injective, since so is  $A_\lambda$ .

As for the equivalence of (i)–(iv), we shall show (ii)  $\Rightarrow$  (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (ii). It is clear that (ii)  $\Rightarrow$  (i) (and also (iv)  $\Rightarrow$  (iii)). We show (i)  $\Rightarrow$  (iii). Assume (i), let  $g \in L^p(\Omega)$  and set  $u := A_\lambda^{-1}g \in W^{2,p}(\Omega) \cap V$ . For all  $v \in V \cap W^{1,q}(\Omega)$ , we have

$$\int_\Omega gv \, dx = \langle A_\lambda u, v \rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv) \, dx = \int_{\Gamma_N} (\partial_\nu u) v \, d\mathcal{H}^{n-1} + \int_\Omega (-\Delta u + \lambda u) v \, dx,$$

which implies that  $u \in X^p$  and  $-\Delta u(x) + \lambda u(x) = g(x)$  for a.e.  $x \in \Omega$ . Hence  $u = B_\lambda^{-1}g$ . Therefore,  $B_\lambda$  is surjective from  $X^p \cap V$  into  $L^p(\Omega)$ . By the open mapping theorem, we obtain  $(-\Delta + \lambda) = B_\lambda \in \text{Isom}(X^p \cap V, L^p(\Omega))$ .

We next show (iii)  $\Rightarrow$  (iv). Under the condition (iii), it holds that  $B_\lambda \in \text{Isom}(X^p \cap V, L^p(\Omega))$ , in particular,  $B_\lambda : X^p \cap V \rightarrow L^p(\Omega)$  is a Fredholm operator of index zero. For arbitrary  $\mu > 0$ , we find that  $B_\mu = B_\lambda + (\mu - \lambda)$  is a Fredholm operator of index zero from  $X^p \cap V$  to  $L^p(\Omega)$  as well, since  $X^p \cap V$  is compactly embedded in  $L^p(\Omega)$ . Since  $B_\mu$  is injective (i.e.,  $\dim \ker(B_\mu) = 0$ ), we infer that  $B_\mu$  is surjective, and hence  $B_\mu$  also belongs to  $\text{Isom}(X^p \cap V, L^p(\Omega))$ . Furthermore,

for any  $g \in L^p(\Omega)$  and  $\mu > 0$ , the element  $u = (-\Delta + \mu)^{-1}g = B_\mu^{-1}g$  belongs to  $X^p \cap V$ . Hence  $B_\mu u = g$ , i.e.,  $A_\mu u = g$ , which implies  $A_\mu^{-1}g = u \in X^p$ . Thus (iv) follows.

It is obvious that (iv) implies (ii) by the definition of  $X^p$ . Thus we have shown that all the conditions (i)–(iv) are equivalent to each other.  $\square$

**Remark 3.7.** Under (28), the assumptions for  $\psi$  in (19) is equivalent to  $\psi \in X^p \cap V$ . Indeed, let  $\psi \in V$  satisfy  $A\psi \in L^p(\Omega)$ . Then as in the proof of Proposition 3.5, one can check that  $\psi \in W^{2,p}(\Omega)$ . Moreover, by Green's formula, we find that

$$\begin{aligned} \int_{\Omega} A\psi v \, dx &= \langle A\psi, v \rangle_V = \int_{\Omega} \nabla \psi \cdot \nabla v \, dx + \sigma \int_{\Omega} \psi v \, dx \\ &= \int_{\Omega} (-\Delta \psi + \sigma \psi) v \, dx + \int_{\Gamma_N} (\partial_\nu \psi) v \, d\mathcal{H}^{n-1} \end{aligned}$$

for all  $v \in V \cap W^{1,q}(\Omega)$ . Thus by the arbitrariness of  $v$ , we obtain  $\partial_\nu \psi = 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\Gamma_N$ , whence follows  $\psi \in X^p$ .

Now, we are in position to state a regularity result for (27) as a corollary of Theorem 3.2. This corollary will be used for proving Theorem 2.6 in Section 5.

**Corollary 3.8** (Regularity of solutions for variational inequalities of obstacle type). *Assume that (13), (18),  $f \in L^p(\Omega)$ ,  $\psi \in X^p \cap V$  and (28) are satisfied. Let  $u \in V$  be the unique element satisfying (a)–(h). Then it holds that*

$$u \in X^p \cap K_0, \quad f \leq Au \leq f \vee \hat{f} \quad \text{a.e. in } \Omega. \quad (29)$$

*Proof.* By Theorem 3.2, the unique element  $u \in V$  satisfying (a)–(h) belongs to  $K_2$ . Hence, by Proposition 3.5, one has  $u \in W^{2,p}(\Omega)$ . Since one can observe that  $u = A_1^{-1}(Au + (1 - \sigma)u)$  and  $Au + (1 - \sigma)u \in L^p(\Omega)$ , by (28) along with Proposition 3.6, it holds that  $u \in X^p$ .  $\square$

We next give a comparison theorem for variational inequalities of obstacle type.

**Theorem 3.9** (Comparison principle for variational inequalities of obstacle type). *We suppose that (13) and (18) are satisfied. For  $i = 1, 2$ , let  $f_i \in L^p(\Omega)$  and  $\psi_i \in V$  be such that  $A\psi_i \in L^p(\Omega)$  and set  $K_0^i := \{v \in V : v \geq \psi_i \text{ a.e. in } \Omega\}$ . Let  $u_i \in V$  be the unique solution of the variational inequality:*

$$u_i \in K_0^i, \quad a(u_i, v - u_i) \geq \langle f_i, v - u_i \rangle_V \quad \text{for all } v \in K_0^i \quad (30)$$

*for  $i = 1, 2$ . If  $f_1 \leq f_2$  and  $\psi_1 \leq \psi_2$  a.e. in  $\Omega$ , then  $u_1 \leq u_2$  a.e. in  $\Omega$ .*

This theorem will be used to prove Theorem 2.8, a comparison theorem for the evolutionary problem (4), with the aid of the discretization argument.

To prove Theorem 3.9, we prepare the following lemma.

**Lemma 3.10.** *We suppose that (13), (18) and (19) are satisfied. Let  $u \in V$  be the unique solution of (a)–(h). Then it holds that  $u \leq w$  a.e. in  $\Omega$  for all  $w \in K_0 \cap K_1$  satisfying  $Aw \in L^p(\Omega)$ .*

*Proof.* We set  $N := \{x \in \Omega : w(x) < u(x)\}$  and  $v := u \wedge w$ . Since  $w \in K_0$ , by Lemma 3.4 and (20),  $v$  satisfies

$$v \in K_0, \quad v = \begin{cases} u & \text{a.e. in } \Omega \setminus N, \\ w & \text{a.e. in } N, \end{cases} \quad \nabla v = \begin{cases} \nabla u & \text{a.e. in } \Omega \setminus N, \\ \nabla w & \text{a.e. in } N. \end{cases}$$

Substituting  $v$  into the variational inequality (b) of Proposition 3.1, we have

$$0 \leq \langle Au - f, v - u \rangle_V = \int_N (Au - f)(w - u) \, dx.$$

Since  $Au - f \geq 0$  and  $w - u < 0$  a.e. in  $N$ , it follows that  $Au = f$  a.e. in  $N$ . Here we note that

$$v - u \in V, \quad v - u = \begin{cases} 0 & \text{a.e. in } \Omega \setminus N, \\ w - u < 0 & \text{a.e. in } N, \end{cases} \quad \nabla(v - u) = \begin{cases} 0 & \text{a.e. in } \Omega \setminus N, \\ \nabla(w - u) & \text{a.e. in } N. \end{cases}$$

From the fact that  $u - v = (u - w)_+$  and  $Aw \in L^p(\Omega)$ , we obtain

$$\begin{aligned} 0 \leq a(u - v, u - v) &= a(u - w, u - v) = \langle A(u - w), u - v \rangle_V \\ &= \int_N (Au - Aw)(u - w) \, dx = \int_N (f - Aw)(u - w) \, dx. \end{aligned}$$

Since  $Aw \geq f$  (by  $w \in K_1$ ) and  $u > w$  a.e. in  $N$ , we conclude that  $a(u - v, u - v) = 0$ , whence  $u = v$  (hence  $u \leq w$ ) a.e. in  $\Omega$ .  $\square$

The lemma above will also play a crucial role for identifying the limit of each solution  $u = u(x, t)$  for (4)–(7) as  $t \rightarrow \infty$  in a proof of Theorem 2.9.

We are now ready to prove Theorem 3.9.

*Proof of Theorem 3.9.* By assumption, we find that  $K_0^2 \subset K_0^1$  and  $K_1^2 \subset K_1^1$ . Moreover,  $u_2$  belongs to both  $K_0^1$  and  $K_1^1 := \{v \in V : Av \geq f_1 \text{ in } V'\}$ , and  $Au_2 \in L^p(\Omega)$  as well. By Lemma 3.10, we conclude that  $u_1 \leq u_2$  a.e. in  $\Omega$ .  $\square$

#### 4. REDUCTION TO AN EVOLUTION EQUATION AND THE UNIQUENESS OF SOLUTION

In this section, we first reduce the problem (4)–(7) to the Cauchy problem for a nonlinear evolution equation in  $L^2(\Omega)$  with the aid of convex analysis. Then we shall prove Theorem 2.4 on the uniqueness of solution.

Let us begin with reformulating (1) as a parabolic *inclusion* with a multivalued nonlinear operator acting on the time derivative of  $u(x, t)$ . Let  $\alpha : \mathbb{R} \rightarrow 2^{\mathbb{R}}$  be given by

$$\alpha(s) = \begin{cases} \{0\} & \text{if } s > 0, \\ (-\infty, 0] & \text{if } s = 0 \end{cases} \quad (31)$$

with the domain  $D(\alpha) = [0, \infty)$ . Then  $s + \alpha(s)$  is the (multi-valued) inverse mapping of the function  $(s)_+$ , and it can be also represented by

$$\alpha(s) = \partial I_{[\cdot \geq 0]}(s) \quad \text{for } s \geq 0,$$

where  $I_{[\cdot \geq 0]}$  denotes the indicator function over the set  $[s \geq 0] := \{s \in \mathbb{R} : s \geq 0\}$  and  $\partial$  means the *subdifferential* in the sense of convex analysis (see, e.g., [15] and also (33) below with  $H = \mathbb{R}$ ). Then (4) can be reformulated as a *doubly nonlinear*-type PDE,

$$\partial_t u + \alpha(\partial_t u) \ni \Delta u + f \quad \text{in } Q. \quad (32)$$

We next reduce the PDE (32) to an *evolution equation*. To this end, define a functional  $\phi : L^2(\Omega) \rightarrow [0, \infty]$  by

$$\phi(v) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx & \text{if } v \in V, \\ +\infty & \text{if } v \in L^2(\Omega) \setminus V \end{cases}$$

with the *effective domain*  $D(\phi) := \{v \in L^2(\Omega) : \phi(v) < +\infty\} = V$ . Then we observe that:

**Lemma 4.1.** *The functional  $\phi$  is convex and lower semicontinuous in  $L^2(\Omega)$ . In particular, if (10) is satisfied, then the subdifferential operator  $\partial\phi$  of  $\phi$  (in  $L^2(\Omega)$ ) is characterized as*

$$D(\partial\phi) = X \cap V, \quad \partial\phi(v) = -\Delta v \quad \text{for } v \in X \cap V,$$

where  $\Delta$  stands for the Laplace operator from  $D(\Delta) = X \cap V$  into  $L^2(\Omega)$  as in Proposition 3.6.

Here let us recall the definition of the *subdifferential operator*  $\partial\varphi : H \rightarrow H$  of a proper, lower semicontinuous and convex functional  $\varphi$  defined on a Hilbert space  $H$ ,

$$\partial\varphi(u) := \{\xi \in H : \varphi(v) - \varphi(u) \geq (\xi, v - u)_H \text{ for all } v \in D(\varphi)\} \quad \text{for } u \in D(\varphi), \quad (33)$$

where  $(\cdot, \cdot)_H$  stands for the inner product in  $H$  and  $D(\varphi) := \{w \in H : \varphi(w) < +\infty\}$ , with domain  $D(\partial\varphi) := \{w \in D(\varphi) : \partial\varphi(w) \neq \emptyset\}$ . It is well known that  $\partial\varphi$  is a (possibly multivalued) maximal monotone operator in  $H$  (see, e.g., [15] for more details).

*Proof of Lemma 4.1.* We note that the restriction  $\phi_0 := \phi|_V$  of  $\phi$  onto  $V$  is Fréchet differentiable and the derivative  $\phi'_0$  of  $\phi_0$  satisfies

$$\langle \phi'_0(u), z \rangle_V = \int_{\Omega} \nabla u \cdot \nabla z \, dx \quad \text{for all } z \in V. \quad (34)$$

Now, let  $u \in D(\partial\phi)$  and  $\xi \in \partial\phi(u) \subset L^2(\Omega)$ . From the definition of subdifferentials, we find that  $\partial\phi(u) \subset \partial\phi_0(u) = \{\phi'_0(u)\}$  for all  $u \in D(\partial\phi) \subset D(\phi) = V$ . Hence  $\xi = \phi'_0(u)$ , i.e.,  $\partial\phi(u) = \{\phi'_0(u)\}$  and  $\phi'_0(u) \in L^2(\Omega)$ ; here and henceforth, we simply write  $\partial\phi(u) = \phi'_0(u)$ . It follows that

$$A_1 u = u + \phi'_0(u) = u + \xi \in L^2(\Omega).$$

Moreover, by (10) along with Proposition 3.6, we deduce that  $u = A_1^{-1}(u + \xi) \in X \cap V$  and that  $u + \xi = A_1 u = -\Delta u + u$ . Therefore we deduce that  $\partial\phi(u) = \phi'_0(u) = -\Delta u$  and  $D(\partial\phi) \subset X \cap V$ . On the other hand, it is clear that  $X \cap V \subset D(\partial\phi)$ , and hence,  $D(\partial\phi) = X \cap V$ .  $\square$

Therefore the initial-boundary value problem for (32) equipped with (5)–(7) can be rewritten as the Cauchy problem for an evolution equation in  $L^2(\Omega)$  of  $u(t) := u(\cdot, t)$ ,

$$\partial_t u(t) + \partial I_{[\cdot, \geq 0]}(\partial_t u(t)) + \partial\phi(u(t)) \ni f(t) \text{ in } L^2(\Omega), \quad 0 < t < T, \quad u(0) = u_0, \quad (35)$$

where  $f(t) := f(\cdot, t)$  and  $\partial I_{[\cdot, \geq 0]}$  denotes the subdifferential operator in  $L^2(\Omega)$  of the functional  $I_{[\cdot, \geq 0]} : L^2(\Omega) \rightarrow [0, \infty]$  defined by

$$I_{[\cdot, \geq 0]}(v) = \begin{cases} 0 & \text{if } v \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise} \end{cases} \quad \text{for } v \in L^2(\Omega).$$

We note that  $\partial I_{[\cdot, \geq 0]}(v) = \alpha(v(\cdot))$  for  $v \in L^2(\Omega)$ , where  $\alpha(\cdot)$  is a multivalued function given by (31), and  $D(\partial I_{[\cdot, \geq 0]}) = \{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$  (see, e.g., [15]).

Here and henceforth, for simplicity, we use the same notation  $I_{[\cdot, \geq 0]}$  for the indicator function over  $[0, +\infty)$  defined on  $\mathbb{R}$  as well as for that over the set  $\{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$  defined on  $L^2(\Omega)$ , unless any confusion may arise. Moreover, the subdifferential operators of the both indicator functions are also denoted by  $\partial I_{[\cdot, \geq 0]}$ .

Strong solutions of (35) are defined as follows:

**Definition 4.2** (Strong solution of (35)). *For given  $f \in L^2(Q_T)$  and  $u_0 \in L^2(\Omega)$ , a function  $u \in C([0, T]; L^2(\Omega))$  is called a strong solution of (4)–(7) on  $[0, T]$ , if the following conditions are satisfied:*

- $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; X \cap V)$ ;
- It holds that

$$\partial_t u(t) + \partial I_{[\cdot, \geq 0]}(\partial_t u(t)) + \partial\phi(u(t)) \ni f(t) \text{ in } L^2(\Omega) \text{ for a.e. } t \in (0, T); \quad (36)$$

- $u(0) = u_0$ ,

where the functionals  $I_{[\cdot, \geq 0]}$  and  $\phi$  on  $L^2(\Omega)$  are defined as above.

**Proposition 4.3** (Equivalence of solutions). *The notion of strong solutions for (35) is equivalent to that for (1)–(7) defined by Definition 2.2.*

*Proof.* Since  $\alpha(s)$  is the inverse mapping of  $s \mapsto (s)_+$ , one observes that (1) is equivalent to (32) at each  $(x, t) \in Q$ . Moreover, due to Lemma 4.1, for each strong solution  $u$  of (35),  $u(x, t)$  satisfies (32) a.e. in  $\Omega \times (0, T)$ . Conversely, let  $u$  be a strong solution of (1) in the sense of Definition 2.2. Then from the regularity condition (ii) of Definition 2.2, the right-hand-side of the inclusion

$$\alpha(\partial_t u) \ni \Delta u + f - \partial_t u$$

belongs to  $L^2(\Omega)$  for a.e.  $t \in (0, T)$ . Moreover, recalling that

$$\partial I_{[\cdot \geq 0]}(v) = \{\xi(\cdot) \in L^2(\Omega) : \xi(x) \in \alpha(v(x)) \text{ for a.e. } x \in \Omega\}$$

(see above), the evolution equation (36) holds in  $L^2(\Omega)$  for a.e.  $t \in (0, T)$ .  $\square$

We next provide a chain-rule for the function  $t \mapsto \phi(u(t))$ , which is derived from a standard theory on subdifferential calculus and which will be used frequently to derive energy estimates in later sections.

**Lemma 4.4.** *We suppose that  $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; X \cap V)$ . Then we have:*

(i) *the function*

$$t \mapsto \phi(u(t)) = \frac{1}{2} \int_{\Omega} |\nabla u(t, x)|^2 dx$$

*belongs to  $W^{1,1}(0, T)$ ;*

(ii) *for a.e.  $t \in (0, T)$ , it holds that*

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx = \frac{d}{dt} \phi(u(t)) = (\partial \phi(u(t)), \partial_t u(t))_{L^2(\Omega)} = - \int_{\Omega} \partial_t u \Delta u dx,$$

*where  $(\cdot, \cdot)_{L^2(\Omega)}$  denotes the inner product of  $L^2(\Omega)$ ;*

(iii)  *$u \in C([0, T]; V)$ .*

*Proof.* Thanks to Lemma 3.3 of [15], the assertions (i) and (ii) follow immediately. Concerning (iii), since  $u$  belongs to  $L^\infty(0, T; V)$  and  $C([0, T]; L^2(\Omega))$ , by exploiting Lemma 8.1 of [32], one finds that  $u$  is continuous on  $[0, T]$  with respect to the weak topology of  $V$ . On the other hand,  $t \mapsto \|u(t)\|_V$  is continuous on  $[0, T]$  by (i). Therefore from the uniform convexity of  $\|\cdot\|_V$ , we deduce that  $t \mapsto u(t)$  is continuous on  $[0, T]$  with respect to the strong topology of  $V$ .  $\square$

Before proceeding to a proof of Theorem 2.4, let us note that

$$|a_+ - b_+|^2 \leq |a_+ - b_+| |a - b| = (a_+ - b_+)(a - b) \quad \text{for all } a, b \in \mathbb{R}, \quad (37)$$

since the function  $s \mapsto s_+ = s \vee 0$  is nondecreasing and non-expansive, that is,  $|a_+ - b_+| \leq |a - b|$ . Now, we are ready to prove Theorem 2.4.

*Proof of Theorem 2.4.* For each  $i = 1, 2$ , let  $u_i$  be a strong solution of (4)–(7) with  $u_0 = u_{0,i} \in V$  and  $f = f_i \in L^2(0, T; L^2(\Omega))$  and set  $u = u_1 - u_2$ . Then due to Lemma 4.4, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx &= - \int_{\Omega} \partial_t u \Delta u dx \\ &= - \int_{\Omega} [(\Delta u_1 + f_1)_+ - (\Delta u_2 + f_2)_+] [(\Delta u_1 + f_1) - (\Delta u_2 + f_2) - f_1 + f_2] dx \\ &\stackrel{(37)}{\leq} - \frac{1}{2} \int_{\Omega} |(\Delta u_1 + f_1)_+ - (\Delta u_2 + f_2)_+|^2 dx \\ &\quad + \frac{1}{2} \int_{\Omega} |f_1 - f_2|^2 dx \quad \text{for a.e. } t \in (0, T), \end{aligned}$$

which implies that

$$\int_{\Omega} |\partial_t u_1 - \partial_t u_2|^2 dx + \frac{d}{dt} \int_{\Omega} |\nabla u_1 - \nabla u_2|^2 dx \leq \int_{\Omega} |f_1 - f_2|^2 dx \quad \text{for a.e. } t \in (0, T).$$



Integrate both sides with respect to  $t$  to obtain

$$\begin{aligned} & \int_0^T \|\partial_t u_1(t) - \partial_t u_2(t)\|_{L^2(\Omega)}^2 dt + \sup_{t \in [0, T]} \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)}^2 \\ & \leq 2 \left( \|\nabla u_{0,1} - \nabla u_{0,2}\|_{L^2(\Omega)}^2 + \int_0^T \|f_1(t) - f_2(t)\|_{L^2(\Omega)}^2 dt \right). \end{aligned} \quad (38)$$

In particular, if  $u_{0,1} = u_{0,2}$  and  $f_1 = f_2$ , then  $u_1$  coincides with  $u_2$  a.e. in  $Q_T$ . Consequently, the solution of (4)–(7) is unique.  $\square$

**Corollary 4.5** (Continuous dependence of solutions on data). *For each  $T > 0$  and  $i = 1, 2$ , let  $u_i$  be the strong solution of (4)–(7) on  $[0, T]$  with  $u_0 = u_{0,i} \in V$  and  $f = f_i \in L^2(Q_T)$ . Then (38) holds true.*

## 5. EXISTENCE OF SOLUTIONS AND COMPARISON PRINCIPLE

In this section, we shall prove Theorem 2.6 on the existence of solutions for (4)–(7). Let  $T > 0$  be fixed. We denote by  $\tau$  a division  $\{t_0, t_1, \dots, t_m\}$  of the interval  $[0, T]$  given by

$$0 = t_0 < t_1 < \dots < t_m = T, \quad \tau_k := t_k - t_{k-1} \quad \text{for } k = 1, \dots, m, \quad |\tau| := \max_{k=1, \dots, m} \tau_k.$$

We shall construct  $u_k \in X \cap V$  (for  $k = 1, 2, \dots, m$ ), which is an approximation of  $u(t_k)$  for a solution  $u$  of (1) by the backward-Euler scheme

$$\frac{u_k - u_{k-1}}{\tau_k} = (\Delta u_k + f_k)_+ \quad \text{a.e. in } \Omega, \quad (39)$$

where  $f_k \in L^2(\Omega)$  is given by

$$f_k := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} f(\cdot, s) ds.$$

For given  $u_0 \in V$ , we shall inductively define  $u_k \in V$  for  $k = 1, 2, \dots, m$  as a (global) minimizer of the functional

$$J_k(v) := \frac{1}{2\tau_k} \int_{\Omega} |v|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \left\langle \frac{u_{k-1}}{\tau_k} + f_k, v \right\rangle_V \quad \text{for } v \in V \quad (40)$$

subject to

$$v \in K_0^k := \{v \in V : v \geq u_{k-1} \text{ a.e. in } \Omega\}. \quad (41)$$

**Remark 5.1** (Derivation of the discretized problems). The minimization problems with constraints stated above can be also derived from a discretization of the evolution equation (35), which is equivalent to (4) (see Remark 2.3 and Section 4). A natural time-discretization of (35) may be given as

$$\frac{u_k - u_{k-1}}{\tau_k} + \partial_V I_{[\cdot \geq 0]} \left( \frac{u_k - u_{k-1}}{\tau_k} \right) - \Delta u_k \ni f_k \quad \text{in } V' \quad (42)$$

(here  $\partial_V$  stands for the subdifferential of the functional  $I_{[\cdot \geq 0]}$  restricted onto  $V$ ), which is an Euler-Lagrange equation for the functional

$$E_k(v) := \frac{1}{2\tau_k} \int_{\Omega} |v|^2 dx + I_{[\cdot \geq 0]} \left( \frac{v - u_{k-1}}{\tau_k} \right) + \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \left\langle \frac{u_{k-1}}{\tau_k} + f_k, v \right\rangle_V \quad \text{for } v \in V.$$

Indeed, since  $E_k$  is coercive, lower semicontinuous and convex in  $V$ ,  $E_k$  admits a global minimizer  $u_k$  over  $V$ , and moreover,  $u_k$  solves (42) in  $V'$ . Here we note that the minimization of  $E_k$  over  $V$  is equivalent to that of  $J_k$  over  $K_0^k$  from the fact that

$$I_{[\cdot \geq 0]} \left( \frac{v - u_{k-1}}{\tau_k} \right) = I_{[\cdot \geq u_{k-1}]}(v) := \begin{cases} 0 & \text{if } v \geq u_{k-1} \text{ a.e. in } \Omega, \\ \infty & \text{otherwise} \end{cases} \quad \text{for } v \in L^2(\Omega).$$

Applying the regularity theory established in Section 3, one can actually obtain the unique minimizer  $u_k$  of  $J_k$  over  $K_0^k$  for each  $k$ . More precisely, we obtain the following theorem, where we set

$$g_k := \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k.$$

**Lemma 5.2** (Existence and regularity of minimizers). *For given  $u_0 \in V$  and each  $k = 1, 2, \dots, m$ , there exists a unique element  $u_k \in K_0^k$  which minimizes (40) subject to (41). Moreover, for each  $k = 1, 2, \dots, m$ , the minimizer  $u_k$  belongs to  $X$  and fulfills (39), that is,*

$$u_k - u_{k-1} \geq 0 \quad \text{a.e. in } \Omega, \quad (43)$$

$$g_k \geq 0 \quad \text{a.e. in } \Omega, \quad (44)$$

$$\langle g_k, u_k - u_{k-1} \rangle_V = 0, \quad (45)$$

Furthermore, one has

$$\langle g_k, v - u_k \rangle_V \geq 0 \quad \text{for all } v \in K_0^k, \quad (46)$$

$$\langle g_k + f_k + \Delta u_{k-1}, v - u_k \rangle_V \geq 0 \quad \text{for all } v \in K_1^k, \quad (47)$$

where  $\Delta$  is the Laplace operator from  $X \cap V$  into  $L^2(\Omega)$  (see Section 4) and the set  $K_1^k$  is given by

$$K_1^k := \left\{ v \in V : \frac{v - u_{k-1}}{\tau_k} - \Delta v - f_k \geq 0 \text{ in } V' \right\}.$$

In addition, suppose that the conditions (18) and (28) hold. If  $u_0 \in X^p \cap V$  and  $\{f_k\}_{k=1}^m \subset L^p(\Omega)$ , then  $\{u_k\}_{k=1}^m \subset X^p$  and it holds that

$$0 \leq g_k \leq (-\Delta u_{k-1} - f_k)_+ \quad \text{a.e. in } \Omega \quad \text{for each } k = 1, \dots, m. \quad (48)$$

*Proof.* Let us start with  $k = 1$ . By setting  $\sigma = 1/\tau_k > 0$  (i.e.,  $Au = A_\sigma u = u/\tau_k - \Delta u$ ),  $f = f_k + u_{k-1}/\tau_k \in L^2(\Omega)$  and  $\psi = u_{k-1} \in X \cap V$ , one can write  $K_0^k = K_0$  and  $J_k(v) = J(v)$  for  $v \in V$  with  $K_0$  and  $J(v)$  defined by (14) and (15) along with (12). Hence, one can apply Proposition 3.1 and Theorem 3.2 to the minimization problem of  $J_k$  over  $K_0^k$ . Then the minimizer  $u_k \in K_0^k$  of  $J_k$  over  $K_0^k$  uniquely exists, and furthermore, (44)–(47) follow immediately from the fact  $u_k \in K_1^k$ , (b), (c) and (d) of Proposition 3.1, respectively. Moreover, by virtue of (10) and Corollary 3.8, one can deduce that  $u_k \in X$ . Repeating the argument above for  $k = 2, 3, \dots, m$ , we can inductively obtain  $u_k \in K_0^k \cap X$  satisfying (44)–(47) for each  $k = 2, 3, \dots, m$ .

Finally, if  $u_0 \in X^p \cap V$ ,  $f_k \in L^p(\Omega)$  and (28) is satisfied for  $p$  satisfying (18), by Corollary 3.8, we can assure that  $u_k \in X^p \cap K_0^k$  and

$$f_k + \frac{u_{k-1}}{\tau_k} \leq \frac{u_k}{\tau_k} - \Delta u_k \leq \left( f_k + \frac{u_{k-1}}{\tau_k} \right) \vee \left( \frac{u_{k-1}}{\tau_k} - \Delta u_{k-1} \right) \quad \text{for a.e. } x \in \Omega,$$

which is equivalent to (48).  $\square$

*Proof of Theorem 2.6.* Let us define the *piecewise linear interpolant*  $u_\tau \in W^{1,\infty}(0, T; X \cap V)$  of  $\{u_k\}$  and the *piecewise constant interpolants*  $\bar{u}_\tau \in L^\infty(0, T; X \cap V)$  and  $\bar{f}_\tau \in L^\infty(0, T; L^2(\Omega))$  of  $\{u_k\}$  and  $\{f_k\}$ , respectively, by

$$\begin{aligned} u_\tau(t) &:= u_{k-1} + \frac{t - t_{k-1}}{\tau_k} (u_k - u_{k-1}) & \text{for } t \in [t_{k-1}, t_k] \text{ and } k = 1, \dots, m, \\ \bar{u}_\tau(t) &:= u_k, \quad \bar{f}_\tau(t) := f_k & \text{for } t \in (t_{k-1}, t_k] \text{ and } k = 1, \dots, m. \end{aligned}$$

By summing up (45) for  $k = 1, \dots, \ell$  with an arbitrary natural number  $\ell \leq m$ , we have

$$\begin{aligned} & \sum_{k=1}^{\ell} \tau_k \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} |\nabla u_{\ell}|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx \\ & \leq \sum_{k=1}^{\ell} \tau_k \left( f_k, \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)} \leq \frac{1}{2} \sum_{k=1}^{\ell} \tau_k \|f_k\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{\ell} \tau_k \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2, \end{aligned} \quad (49)$$

which implies

$$\int_0^t \|\partial_t u_{\tau}(s)\|_{L^2(\Omega)}^2 ds + \|\nabla \bar{u}_{\tau}(t)\|_{L^2(\Omega)}^2 \leq \|\nabla u_0\|_{L^2(\Omega)}^2 + \int_0^t \|\bar{f}_{\tau}(s)\|_{L^2(\Omega)}^2 ds \quad \text{for all } t \in [0, T].$$

Hence, we obtain

$$\begin{aligned} & \|\partial_t u_{\tau}\|_{L^2(0, T; L^2(\Omega))}^2 + \sup_{t \in [0, T]} \|\nabla \bar{u}_{\tau}(t)\|_{L^2(\Omega)}^2 + \sup_{t \in [0, T]} \|\nabla u_{\tau}(t)\|_{L^2(\Omega)}^2 \\ & \leq C \left( \|\nabla u_0\|_{L^2(\Omega)}^2 + \|\bar{f}_{\tau}\|_{L^2(0, T; L^2(\Omega))}^2 \right). \end{aligned} \quad (50)$$

Now, let us take a limit as  $m \rightarrow \infty$  such that  $|\tau| \rightarrow 0$  and note that

$$\bar{f}_{\tau} \rightarrow f \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (51)$$

In particular,  $\{\bar{f}_{\tau}\}$  is bounded in  $L^2(0, T; L^2(\Omega))$ . Indeed, one can verify that

$$\|\bar{f}_{\tau}\|_{L^2(0, T; L^2(\Omega))} \leq \|f\|_{L^2(0, T; L^2(\Omega))}.$$

From the uniform estimate (50), one can take a function  $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^{\infty}(0, T; V)$  (in particular,  $u \in C([0, T]; L^2(\Omega))$  as well) such that, up to a (non-relabelled) subsequence,

$$u_{\tau} \rightarrow u \quad \text{weakly in } W^{1,2}(0, T; L^2(\Omega)), \quad (52)$$

$$\text{weakly star in } L^{\infty}(0, T; V), \quad (53)$$

$$\text{strongly in } C([0, T]; L^2(\Omega)), \quad (54)$$

$$\bar{u}_{\tau} \rightarrow u \quad \text{weakly star in } L^{\infty}(0, T; V), \quad (55)$$

$$u_{\tau}(T) \rightarrow u(T) \quad \text{weakly in } V. \quad (56)$$

Here, the weak and weak star convergence of  $u_{\tau}$  and  $\bar{u}_{\tau}$  immediately follow from the uniform estimate (50). Moreover, we also note that  $u_{\tau}$  and  $\bar{u}_{\tau}$  possess a common limit function. Indeed, by a simple calculation, we observe that

$$\begin{aligned} \|u_{\tau}(t) - \bar{u}_{\tau}(t)\|_{L^2(\Omega)} &= \left| \frac{t_k - t}{\tau_k} \right| \|u_k - u_{k-1}\|_{L^2(\Omega)} \\ &\leq \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)} \tau_k \\ &\stackrel{(49)}{\leq} C |\tau|^{1/2} \quad \text{for all } t \in (t_{k-1}, t_k], \quad k = 1, 2, \dots, m, \end{aligned}$$

which yields that

$$\sup_{t \in [0, T]} \|u_{\tau}(t) - \bar{u}_{\tau}(t)\|_{L^2(\Omega)} \leq C |\tau|^{1/2} \rightarrow 0.$$

Thus  $u_{\tau}$  and  $\bar{u}_{\tau}$  (weakly) converge to a common limit function. Furthermore, since  $V$  is compactly embedded in  $L^2(\Omega)$ , due to Ascoli's compactness lemma along with (50), we obtain the strong convergence (54). Since  $u_{\tau}(T) = u_m$  is bounded in  $V$  by (49), one can also derive (56) from (54). We further observe that  $u(0) = u_0$ .

We next estimate  $\Delta \bar{u}_{\tau}$  in  $L^2(0, T; L^2(\Omega))$  by using (48) and the assumption (11). We first rewrite (48) as

$$-\frac{u_k - u_{k-1}}{\tau_k} + f_k \leq -\Delta u_k \leq \left( -\frac{u_k - u_{k-1}}{\tau_k} + f_k \right) \vee \left( -\frac{u_k - u_{k-1}}{\tau_k} - \Delta u_{k-1} \right) \quad \text{a.e. in } \Omega. \quad (57)$$

Since  $(u_k - u_{k-1})/\tau_k \geq 0$  a.e. in  $\Omega$  by  $u_k \in K_0^k$ , we observe by (11) that

$$(\text{The right-hand side of (57)}) \leq f_k \vee (-\Delta u_{k-1}) \leq f^* \vee (-\Delta u_{k-1}) \quad \text{a.e. in } \Omega,$$

which also iteratively implies that

$$\begin{aligned} -\Delta u_k &\leq f^* \vee (-\Delta u_{k-1}) \\ &\leq f^* \vee (f^* \vee (-\Delta u_{k-2})) \\ &= f^* \vee (-\Delta u_{k-2}) \leq \cdots \leq f^* \vee (-\Delta u_0) \quad \text{a.e. in } \Omega. \end{aligned}$$

Thus we obtain

$$-\frac{u_k - u_{k-1}}{\tau_k} + f_k \leq -\Delta u_k \leq f^* \vee (-\Delta u_0) \quad \text{a.e. in } \Omega,$$

which yields that

$$\|\Delta u_k\|_{L^2(\Omega)}^2 \leq 2 \left( \|f^*\|_{L^2(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2 + \|f_k\|_{L^2(\Omega)}^2 \right)$$

for  $k = 1, 2, \dots, m$ . Hence we deduce that

$$\begin{aligned} \int_0^T \|\Delta \bar{u}_\tau(t)\|_{L^2(\Omega)}^2 dt &\leq 2T \left( \|f^*\|_{L^2(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 \right) \\ &\quad + 2 \int_0^T \|\partial_t u_\tau(t)\|_{L^2(\Omega)}^2 dt + 2 \int_0^T \|\bar{f}_\tau(t)\|_{L^2(\Omega)}^2 dt \leq C \end{aligned} \quad (58)$$

by using (50) and (51).

Exploiting Proposition 3.6 with (10), we see that  $(I - \Delta) \in \text{Isom}(X \cap V, L^2(\Omega))$ , which together with (58) gives

$$\int_0^T \|\bar{u}_\tau(t)\|_X^2 dt \leq C \int_0^T \left( \|\Delta \bar{u}_\tau(t)\|_{L^2(\Omega)}^2 + \|\bar{u}_\tau(t)\|_{L^2(\Omega)}^2 \right) dt \leq C.$$

Therefore we have, up to a (non-relabelled)subsequence,

$$\begin{aligned} \bar{u}_\tau &\rightarrow u \quad \text{weakly in } L^2(0, T; X), \\ \Delta \bar{u}_\tau &\rightarrow \Delta u \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \end{aligned}$$

which particularly implies  $u(t) \in D(\Delta) = X \cap V$  for a.e.  $t \in (0, T)$ . Therefore the piecewise constant interpolant  $\bar{g}_\tau$  of  $\{g_k\}$  defined by

$$\bar{g}_\tau(t) := g_k \stackrel{(44)}{=} \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k \quad \text{for } t \in (t_{k-1}, t_k]$$

converges to

$$\partial_t u - \Delta u - f =: g \quad (59)$$

weakly in  $L^2(0, T; L^2(\Omega))$ .

It remains to prove that  $u$  solves (4) for a.e.  $(x, t) \in Q_T$ . To this end, we recall the evolution equation (35) equivalent to (4). Then it suffices to check that

$$\partial_t u \geq 0 \quad \text{a.e. in } Q_T \quad \text{and} \quad -g(t) \in \partial I_{[\cdot \geq 0]}(\partial_t u(t)) \quad \text{for a.e. } t \in (0, T).$$

To this end, we employ the so-called Minty's trick for maximal monotone operators, since  $\partial I_{[\cdot \geq 0]}$  is maximal monotone in  $L^2(\Omega)$ .

**Proposition 5.3** (Demiclosedness of maximal monotone operators (see, e.g., [15, 16, 10])). *Let  $A : H \rightarrow H$  be a (possibly multivalued) maximal monotone operator defined on a Hilbert space  $H$  equipped with a inner product  $(\cdot, \cdot)_H$ . Let  $[u_n, \xi_n]$  be in the graph of  $A$  such that  $u_n \rightarrow u$  weakly in  $H$  and  $\xi_n \rightarrow \xi$  weakly in  $H$ . Suppose that*

$$\limsup_{n \rightarrow +\infty} (\xi_n, u_n)_H \leq (\xi, u)_H.$$

Then  $[u, \xi]$  belongs to the graph of  $A$ , and moreover, it holds that

$$\lim_{n \rightarrow +\infty} (\xi_n, u_n)_H = (\xi, u)_H.$$

Note that  $(u_k - u_{k-1})/\tau_k \geq 0$  a.e. in  $\Omega$ . For an arbitrary  $w \in D(I_{[\cdot \geq 0]}) = \{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$ , substitute  $v = w\tau_k + u_{k-1} \in K_0^k$  to (46). Then we see that

$$0 \stackrel{(46)}{\geq} \langle -g_k, v - u_k \rangle_V = \tau_k \left( -g_k, w - \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)},$$

which together with the arbitrariness of  $w \in D(I_{[\cdot \geq 0]})$  and the definition of  $I_{[\cdot \geq 0]}$  implies that

$$-g_k \in \partial I_{[\cdot \geq 0]} \left( \frac{u_k - u_{k-1}}{\tau_k} \right), \quad \text{i.e., } -\bar{g}_\tau(t) \in \partial I_{[\cdot \geq 0]} (\partial_t u_\tau(t)).$$

Moreover, for  $k = 1, 2, \dots, m$ , we find by (44) that

$$\left( -g_k, \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)} \leq - \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2 - \frac{\phi(u_k) - \phi(u_{k-1})}{\tau_k} + \left( f_k, \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)},$$

which leads us to get

$$\begin{aligned} \int_0^T (-\bar{g}_\tau(t), \partial_t u_\tau(t))_{L^2(\Omega)} dt &\leq - \int_0^T \|\partial_t u_\tau(t)\|_{L^2(\Omega)}^2 dt - \phi(u_\tau(T)) + \phi(u_0) \\ &\quad + \int_0^T (\bar{f}_\tau(t), \partial_t u_\tau(t))_{L^2(\Omega)} dt. \end{aligned}$$

Taking a limsup as  $|\tau| \rightarrow 0$  in both sides, exploiting the weak lower semicontinuity of norms and the functional  $\phi(\cdot)$ , and recalling Lemma 4.4, we conclude that

$$\begin{aligned} \limsup_{|\tau| \rightarrow 0} \int_0^T (-\bar{g}_\tau(t), \partial_t u_\tau(t))_{L^2(\Omega)} dt &\leq - \int_0^T \|\partial_t u(t)\|_{L^2(\Omega)}^2 dt - \phi(u(T)) + \phi(u_0) \\ &\quad + \int_0^T (f(t), \partial_t u(t))_{L^2(\Omega)} dt \\ &= \int_0^T (-\partial_t u(t) + \Delta u(t) + f(t), \partial_t u(t))_{L^2(\Omega)} dt \\ &\stackrel{(59)}{=} \int_0^T (-g(t), \partial_t u(t))_{L^2(\Omega)} dt. \end{aligned} \tag{60}$$

Consequently, by virtue of the (weak) closedness of maximal monotone operators (see Proposition 5.3 above), it follows that  $\partial_t u(t) \in D(\partial I_{[\cdot \geq 0]})$ , i.e.,  $\partial_t u(t) \geq 0$  a.e. in  $\Omega$ , and  $-g(t) \in \partial I_{[\cdot \geq 0]}(\partial_t u(t))$  for a.e.  $t \in (0, T)$ . Therefore  $u$  solves (35), and hence,  $u$  is a strong solution of (4)–(7). Thus Theorem 2.6 has been proved.  $\square$

**Remark 5.4.** To prove that  $u$  is a strong solution of (1)–(7), it is possible to show the conditions (V1)–(V6) of Theorem 7.1 in §7, instead of the last argument of the proof of Theorem 2.6. Actually, (V2) and (V3) directly follow from (43) and (44) by taking limit of  $|\tau| \rightarrow 0$ , respectively. The condition (V4) follows from the estimate (60), since the left-hand side of (60) is zero and the right-hand side is non-positive.

Due to Theorem 2.4, the limit of  $\{u_\tau\}$  and  $\{\bar{u}_\tau\}$  is unique, whence they converge along the full sequence.

**Corollary 5.5.** Sequences  $\{u_\tau\}$  and  $\{\bar{u}_\tau\}$  converge to the unique solution  $u$  of (1)–(7) as  $|\tau| \rightarrow 0_+$ .

We next prove Theorem 2.8.

*Proof of Theorem 2.8.* Let  $u^1$  and  $u^2$  be strong solutions of (4) with  $u_0 = u_0^i$  and  $f = f^i$  for  $i = 1, 2$ , respectively. By the uniqueness of solutions (see Theorem 2.4) and the construction of solutions discussed so far, one can take discretized solutions  $\{u_k^i\}$  for  $i = 1, 2$  such that the piecewise linear interpolant  $u_\tau^i$  of  $\{u_k^i\}$  converges to  $u^i$  strongly in  $C([0, T]; L^2(\Omega))$  as  $|\tau| \rightarrow 0$ , and they solve the variational inequalities

$$\begin{cases} u_k^i \in K_0(u_{k-1}^i) := \{v \in V : v \geq u_{k-1}^i \text{ a.e. in } \Omega\}, \\ a_k(u_k^i, v - u_k^i) \geq \int_{\Omega} (f_k^i + u_{k-1}^i/\tau_k) (v - u_k^i) dx \quad \text{for all } v \in K_0(u_{k-1}^i), \end{cases}$$

where  $a_k(\cdot, \cdot)$  stands for the bilinear form given by

$$a_k(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \frac{1}{\tau_k} \int_{\Omega} uv dx \quad \text{for } u, v \in V.$$

By iteratively applying the comparison theorem for elliptic variational inequalities (see Theorem 3.9), from the fact that  $f^1 \leq f^2$  a.e. in  $Q_T$  and  $u_0^1 \leq u_0^2$  a.e. in  $\Omega$ , one can deduce that

$$u_k^1 \leq u_k^2 \text{ a.e. in } Q_T \quad \text{for all } k = 1, 2, \dots, m,$$

which also implies  $u_\tau^1(t) \leq u_\tau^2(t)$  a.e. in  $\Omega$  for all  $t \in (0, T)$ . Then passing to the limit as  $|\tau| \rightarrow 0$ , we conclude that  $u^1 \leq u^2$  a.e. in  $Q_T$ .  $\square$

## 6. LONG-TIME BEHAVIOR OF SOLUTIONS

This section is devoted to proving Theorem 2.9. Let us begin with deriving a uniform estimate for  $u(t)$  for  $t \geq 0$ . To do so, recall the construction of the unique solution  $u = u(x, t)$  of (4)–(7) performed in the proof of Theorem 2.6 and particularly note that

$$u_k \geq u_{k-1} \quad \text{and} \quad g_k := \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k \geq 0 \quad \text{a.e. in } \Omega. \quad (61)$$

It follows that

$$h_k := g_k + f_k = \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k \geq -\Delta u_k \quad \text{a.e. in } \Omega. \quad (62)$$

We also recall the estimate (48), which gives

$$f_k \leq h_k \leq (-\Delta u_{k-1}) \vee f_k \quad \text{a.e. in } \Omega. \quad (63)$$

Therefore by (H3) we find that

$$\begin{aligned} f_k &\stackrel{(63)}{\leq} h_k \stackrel{(63)}{\leq} (-\Delta u_{k-1}) \vee f_k \\ &\stackrel{(62)}{\leq} h_{k-1} \vee f^* \\ &\leq (h_{k-2} \vee f^*) \vee f^* \\ &= h_{k-2} \vee f^* \leq \dots \leq h_1 \vee f^* \stackrel{(63)}{\leq} (-\Delta u_0) \vee f^* \quad \text{a.e. in } \Omega, \end{aligned}$$

which together with the assumption that  $u_0 \in X \cap V$  gives

$$\begin{aligned} \|h_k\|_{L^2(\Omega)} &\leq \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f_k\|_{L^2(\Omega)} \\ &\leq \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f\|_{L^\infty(0, \infty; L^2(\Omega))}. \end{aligned} \quad (64)$$

Here we used the fact that  $\|f_k\|_{L^2(\Omega)} \leq \|f\|_{L^\infty(0, \infty; L^2(\Omega))}$  for all  $k$ . Moreover, set

$$\bar{h}_\tau(t) := \partial_t u_\tau(t) - \Delta \bar{u}_\tau(t) = h_k \quad \text{for } t \in (t_{k-1}, t_k].$$

Recalling the convergence of approximate solutions obtained in the proof of Theorem 2.6, we observe that

$$\bar{h}_\tau \rightarrow \partial_t u - \Delta u =: h \quad \text{weakly in } L^2(0, T; L^2(\Omega)),$$



On the other hand, since  $\{\bar{h}_\tau\}$  is bounded in  $L^\infty(0, T; L^2(\Omega))$  by (64), we assure, up to a (non-relabeled) subsequence, that

$$\bar{h}_\tau \rightarrow h \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega))$$

as  $|\tau| \rightarrow 0$ . Moreover, from the lower semicontinuity of the  $L^\infty$ -norm in the weak star topology, we have, by (64),

$$\|h\|_{L^\infty(0, T; L^2(\Omega))} \leq \liminf_{\tau \rightarrow 0} \|\bar{h}_\tau\|_{L^\infty(0, T; L^2(\Omega))} \leq \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f\|_{L^\infty(0, \infty; L^2(\Omega))}$$

for each  $T > 0$ . Since the bound is independent of  $T > 0$ , one can derive that

$$\|h\|_{L^\infty(0, \infty; L^2(\Omega))} \leq \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f\|_{L^\infty(0, \infty; L^2(\Omega))}. \quad (65)$$

Note that  $(\xi, v)_{L^2(\Omega)} = 0$  for all  $v \in L^2(\Omega)$  with  $v \geq 0$  a.e. in  $\Omega$  and  $\xi \in \partial I_{[\cdot \geq 0]}(v)$ . Thus testing (35) by  $\partial_t u(t)$ , we have

$$\begin{aligned} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2(\Omega)}^2 &= (f(t), \partial_t u(t))_{L^2(\Omega)} \\ &= (f(t) - f_\infty, \partial_t u(t))_{L^2(\Omega)} + \frac{d}{dt} (f_\infty, u(t))_{L^2(\Omega)} \\ &\leq \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|f(t) - f_\infty\|_{L^2(\Omega)}^2 + \frac{d}{dt} (f_\infty, u(t))_{L^2(\Omega)}. \end{aligned}$$

Define an energy functional  $E$  on  $V$  by

$$E(v) := \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - (f_\infty, v)_{L^2(\Omega)} \quad \text{for } v \in V.$$

Then one has

$$\frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{d}{dt} E(u(t)) \leq \frac{1}{2} \|f(t) - f_\infty\|_{L^2(\Omega)}^2 \quad \text{for a.e. } t \geq 0, \quad (66)$$

which implies the non-increase of the function

$$t \mapsto E(u(t)) - \frac{1}{2} \int_0^t \|f(\tau) - f_\infty\|_{L^2(\Omega)}^2 d\tau \quad \text{for } t \geq 0.$$

Moreover, by using the Poincaré inequality (due to (H1)), we have

$$E(v) \geq \frac{1}{4} \|\nabla v\|_{L^2(\Omega)}^2 - C \|f_\infty\|_{L^2(\Omega)}^2 \quad \text{for all } v \in V. \quad (67)$$

Thus integrating (66) over  $(0, s)$  and using (H2) and (67) we obtain

$$\int_0^\infty \|\partial_t u(t)\|_{L^2(\Omega)}^2 dt \leq C, \quad (68)$$

$$\sup_{t \geq 0} \|\nabla u(t)\|_{L^2(\Omega)} \leq C, \quad (69)$$

which also yields

$$\sup_{t \geq 0} \|\Delta u(t)\|_{V'} \leq C. \quad (70)$$

Moreover, by virtue of (65), we have

$$\|g\|_{L^\infty(0, \infty; L^2(\Omega))} \leq \|h\|_{L^\infty(0, \infty; L^2(\Omega))} + \|f\|_{L^\infty(0, \infty; L^2(\Omega))} \leq M \quad (71)$$

for some constant  $M$ . Here we used that  $g(t) = h(t) - f(t)$ .

Let  $I \subset (0, \infty)$  be the set of all  $t \geq 0$  for which (35) holds true and  $\|g(t)\|_{L^2(\Omega)}$  is bounded by  $M$  as in (71). Then the set  $(0, \infty) \setminus I$  has zero Lebesgue measure. Recalling by (68) and (H2) that

$$\int_0^\infty \left( \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \|f(t) - f_\infty\|_{L^2(\Omega)}^2 \right) dt < \infty,$$

one can take a sequence  $s_n \in [n, n+1] \cap I$  such that

$$\partial_t u(s_n) \rightarrow 0 \quad \text{strongly in } L^2(\Omega), \quad (72)$$

$$f(s_n) \rightarrow f_\infty \quad \text{strongly in } L^2(\Omega) \quad (73)$$

as  $n \rightarrow \infty$ .

Moreover, by using the preceding uniform (in  $t$ ) estimates and using the compact embedding  $V \hookrightarrow L^2(\Omega)$ , we deduce, up to a (non-relabelled) subsequence, that

$$u(s_n) \rightarrow z \quad \text{weakly in } V, \quad (74)$$

$$\text{strongly in } L^2(\Omega), \quad (75)$$

$$-\Delta u(s_n) \rightarrow -\Delta z \quad \text{weakly in } V', \quad (76)$$

$$-g(t) \rightarrow \xi \quad \text{weakly in } L^2(\Omega) \quad (77)$$

with some  $z \in V$  and  $\xi \in L^2(\Omega)$ . From the demiclosedness of  $\partial I_{[\cdot, \geq 0]}$  in  $L^2(\Omega)$  and the fact that  $-g(t) \in \partial I_{[\cdot, \geq 0]}(\partial_t u(t))$  for a.e.  $t \in (0, \infty)$ , it follows that  $\xi \in \partial I_{[\cdot, \geq 0]}(0)$ , that is,  $\xi \leq 0$  a.e. in  $\Omega$  by  $\partial I_{[\cdot, \geq 0]}(0) = (-\infty, 0]$ . Moreover, by (35) and (73), we get  $\xi - \Delta z = f_\infty$ , which leads us to  $f_\infty + \Delta z = \xi \leq 0$  a.e. in  $\Omega$ . Furthermore, by (10) along with Proposition 3.6, the limit  $z$  belongs to  $X$ , since  $z - \Delta z = z - \xi + f_\infty \in L^2(\Omega)$ .

Therefore we derive that

$$\begin{aligned} \|\nabla u(s_n)\|_{L^2(\Omega)}^2 &= (-\Delta u(s_n), u(s_n))_{L^2(\Omega)} \\ &= (-\partial_t u(s_n), u(s_n))_{L^2(\Omega)} + (g(s_n), u(s_n))_{L^2(\Omega)} + (f(s_n), u(s_n))_{L^2(\Omega)} \\ &\rightarrow (-\xi + f_\infty, z)_{L^2(\Omega)} = (-\Delta z, z)_{L^2(\Omega)} = \|\nabla z\|_{L^2(\Omega)}^2. \end{aligned}$$

From the uniform convexity of  $V$ , it holds that

$$u(s_n) \rightarrow z \quad \text{strongly in } V. \quad (78)$$

We shall next verify the convergence of the solution  $u(t)$  to the same limit  $z$  as  $t \rightarrow \infty$ , that is,  $u(t_n) \rightarrow z$  for any sequence  $t_n \rightarrow \infty$  and the limit  $z$  is independent of the choice of the sequence  $(t_n)$ . Subtracting the stationary equation

$$\partial I_{[\cdot, \geq 0]}(0) - \Delta z \ni f_\infty$$

from the evolution equation (35), we see that

$$\partial_t u(t) + \partial I_{[\cdot, \geq 0]}(\partial_t u(t)) - \partial I_{[\cdot, \geq 0]}(0) - \Delta(u(t) - z) \ni f(t) - f_\infty.$$

Test it by  $\partial_t u(t)$  to get

$$\frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla(u(t) - z)\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|f(t) - f_\infty\|_{L^2(\Omega)}^2.$$

Integrate both sides over  $(s_n, \tau)$  for  $\tau > s_n$ . Then it follows from (78) that

$$\begin{aligned} \frac{1}{2} \sup_{\tau \geq s_n} \|\nabla(u(\tau) - z)\|_{L^2(\Omega)}^2 &\leq \frac{1}{2} \|\nabla(u(s_n) - z)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{s_n}^{\infty} \|f(t) - f_\infty\|_{L^2(\Omega)}^2 dt \\ &\stackrel{(H2)}{\rightarrow} 0. \end{aligned}$$

Thus  $u(t)$  converges to the limit  $z$  strongly in  $V$  as  $t \rightarrow \infty$ . This completes the proof of the first half of the assertion.

We next prove the second half of the assertion. In addition, assume that  $f(x, t) \leq f_\infty(x)$  for a.e.  $x \in Q$  and let  $\bar{z} \in X \cap V$  be the unique solution of the variational inequality (VI)( $u_0, f_\infty$ ). Then by Proposition 3.1 and Theorem 3.2 for  $A = A_\sigma$  with  $\sigma = 0$ ,  $\bar{z}$  satisfies  $-\Delta \bar{z} \geq f_\infty$  a.e. in  $\Omega$ , and moreover, we deduce that  $U(x, t) := \bar{z}(x)$  becomes a strong solution of (4) by observing that

$$\partial_t U \equiv 0 \quad \text{and} \quad \Delta U(x, t) + f(x, t) \leq \Delta \bar{z}(x) + f_\infty(x) \leq 0 \quad \text{a.e. in } Q.$$

Hence by the comparison principle for the evolutionary problem (4) (see Theorem 2.8), we assure that  $u(x, t) \leq \bar{z}(x)$  for a.e.  $(x, t) \in Q$ . Letting  $t \rightarrow \infty$  and recalling (75), we obtain

$$z(x) \leq \bar{z}(x) \quad \text{for a.e. } x \in \Omega.$$

On the other hand, since  $z$  belongs to  $X \cap V$  and satisfies  $z \geq u_0$  and  $-\Delta z \geq f_\infty$  in  $V'$ , applying the comparison theorem for variational inequalities of obstacle type (see Lemma 3.10) to (VI)( $u_0, f_\infty$ ), we assure that  $\bar{z} \leq z$  a.e. in  $\Omega$ . Consequently, we conclude that  $z = \bar{z}$  a.e. in  $\Omega$ . Thus we have proved the second half of the assertion of Theorem 2.9.  $\square$

## 7. OTHER EQUIVALENT FORMULATIONS

In this section, we discuss other formulations of solutions for (4)–(7) equivalent to those defined by Definition 2.2. Let us start with a *complementarity form* of strong solutions.

**Theorem 7.1.** *Let  $f \in L^2(Q_T)$  and  $u_0 \in L^2(\Omega)$ . Then  $u$  is a strong solution of the problem (4)–(7) on  $[0, T]$ , if and only if the following six conditions are satisfied:*

- (V1)  $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1(\Omega))$ ,
- (V2)  $\partial_t u \geq 0$  a.e. in  $Q_T$ ,
- (V3)  $\partial_t u - \Delta u - f \geq 0$  a.e. in  $Q_T$ ,
- (V4)  $(\partial_t u - \Delta u - f) \partial_t u = 0$  a.e. in  $Q_T$ ,
- (V5)  $\partial_\nu u = 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\Gamma_N$  and  $u = 0$   $\mathcal{H}^{n-1}$ -a.e. on  $\Gamma_D$  for a.e.  $t \in (0, T)$ ,
- (V6)  $u(0, \cdot) = u_0$ .

*Proof.* If  $u$  satisfies (V1) (or (i) of Definition 2.2), one can define the following measurable subsets of  $Q_T$ :

$$\begin{aligned} Q_0 &:= \{(x, t) \in Q_T : \partial_t u \neq (\Delta u + f)_+\}, \\ Q_1 &:= \{(x, t) \in Q_T : \partial_t u = (\Delta u + f)_+ > 0\}, \\ Q_2 &:= \{(x, t) \in Q_T : \partial_t u = (\Delta u + f)_+ = 0\}, \end{aligned}$$

which are disjoint and satisfy  $Q_T = Q_0 \cup Q_1 \cup Q_2$ .

Let  $u$  satisfy (i)–(iii) of Definition 2.2. Conditions (V1), (V5) and (V6) follow immediately. From (ii) of Definition 2.2, it follows that

$$\mathcal{H}^{n+1}(Q_0) = 0,$$

and moreover, by definition,

$$\begin{aligned} \partial_t u &> 0, \quad \partial_t u - \Delta u - f = 0 \quad \text{a.e. in } Q_1, \\ \partial_t u &= 0, \quad \partial_t u - \Delta u - f \geq 0 \quad \text{a.e. in } Q_2. \end{aligned}$$

Hence (V2), (V3) and (V4) follows. Consequently, every strong solution  $u$  of (4)–(7) in the sense of Definition 2.2 satisfies all the conditions (V1)–(V6).

Conversely, let  $u$  satisfy (V1)–(V6). Conditions (i) and (iii) of Definition 2.2 follow from (V1), (V5) and (V6). Let us next show that  $\mathcal{H}^{n+1}(Q_0) = 0$ , which is equivalent to the condition (ii) of Definition 2.2. Define

$$Q^* := \{(x, t) \in Q_T : \partial_t u \geq 0 \text{ and } \partial_t u - \Delta u - f \geq 0 \text{ at } (x, t)\}.$$

Then it holds that  $\mathcal{H}^{n+1}(Q_T \setminus Q^*) = 0$  by (V2) and (V3).

We claim that

$$\partial_t u > 0, \quad \partial_t u - \Delta u - f > 0 \quad \text{at each } (x, t) \in Q_0 \cap Q^*. \quad (79)$$

Indeed, by the definitions of  $Q_0$  and  $Q^*$ ,  $u$  satisfies the following conditions at each  $(x, t) \in Q_0 \cap Q^*$ :

$$\partial_t u \neq (\Delta u + f)_+, \quad (80)$$

$$\partial_t u \geq 0, \quad (81)$$

$$\partial_t u - \Delta u - f \geq 0. \quad (82)$$

If  $\partial_t u = 0$  at some  $(x_0, t_0) \in Q_0 \cap Q^*$ , then  $\Delta u + f > 0$  at  $(x_0, t_0)$  by (80) and it contradicts (82). Hence, we obtain  $\partial_t u > 0$  at each point of  $Q_0 \cap Q^*$  by (81). Similarly, if  $\partial_t u - \Delta u - f = 0$  at some  $(x_0, y_0) \in Q_0 \cap Q^*$ , then  $0 < \partial_t u = \Delta u + f = (\Delta u + f)_+$  at  $(x_0, y_0)$ , which contradicts (80). Thus, we obtain  $\partial_t u - \Delta u - f > 0$  in  $Q_0 \cap Q^*$  by (82).

Since  $Q_T = Q_0 \cup Q_1 \cup Q_2$  is a disjoint union and  $\mathcal{H}^{n-1}(Q_T \setminus Q^*) = 0$ , we have

$$\begin{aligned} 0 &\stackrel{(V4)}{=} \iint_{Q_T} (\partial_t u - \Delta u - f) \partial_t u \, dx \, dt = \iint_{Q_0} (\partial_t u - \Delta u - f) \partial_t u \, dx \, dt \\ &= \iint_{Q_0 \cap Q^*} (\partial_t u - \Delta u - f) \partial_t u \, dx \, dt. \end{aligned}$$

By (79), we obtain  $\mathcal{H}^{n+1}(Q_0 \cap Q^*) = 0$ ; otherwise the last integral is positive. Hence we conclude that

$$\mathcal{H}^{n+1}(Q_0) = \mathcal{H}^{n+1}(Q_0 \cap Q^*) + \mathcal{H}^{n+1}(Q_0 \setminus Q^*) = 0.$$

This completes the proof.  $\square$

Finally, let us discuss a possible formulation of (4) in the sense of viscosity solutions. Set

$$F(x, t, Y) := -(\operatorname{tr} Y + f(x, t))_+ \quad \text{for } x \in \Omega, \, t \in (0, T), \, Y \in \mathbb{R}_{\text{sym}}^{n \times n}, \quad (83)$$

where  $\mathbb{R}_{\text{sym}}^{n \times n}$  denotes the set of all symmetric  $n \times n$  real matrices. Then (1) is also written as

$$\partial_t u(x, t) + F(x, t, D^2 u(x, t)) = 0,$$

where  $D^2 u(x, t) \in \mathbb{R}_{\text{sym}}^{n \times n}$  is the Hessian matrix of  $u$ . Since  $F$  is degenerate elliptic, one may apply the theory of viscosity solutions to prove the existence and uniqueness of viscosity solutions of (83) under suitable assumptions for  $f(x, t)$  and the boundary condition. However, to the authors' knowledge, no result on such a viscosity approach to (4) has been obtained except for [44]. Moreover, the relation between the notion of viscosity solutions and that of strong solutions for (4) is widely open. For further details of the theory of viscosity solutions, we refer the reader to [19], [31], [23] and references therein.

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